

University of Warwick institutional repository: <http://go.warwick.ac.uk/wrap>

**A Thesis Submitted for the Degree of PhD at the University of Warwick**

<http://go.warwick.ac.uk/wrap/77123>

This thesis is made available online and is protected by original copyright.

Please scroll down to view the document itself.

Please refer to the repository record for this item for information to help you to cite it. Our policy information is available from the repository home page.



# Recognising Mapping Classes

by

Mark Christopher Bell

**Thesis**

Submitted to the University of Warwick

for the degree of

**Doctor of Philosophy**

**Warwick Mathematics Institute**

June 2015

THE UNIVERSITY OF  
**WARWICK**

# Contents

|   |             |
|---|-------------|
| <b>List of Tables</b>                       | <b>vii</b>  |
| <b>List of Figures</b>                      | <b>viii</b> |
| <b>Acknowledgments</b>                      | <b>x</b>    |
| <b>Declarations</b>                         | <b>xii</b>  |
| <b>Abstract</b>                             | <b>xiii</b> |
| <b>Abbreviations</b>                        | <b>xiv</b>  |
| <b>Chapter 1 Introduction</b>               | <b>1</b>    |
| 1.1 Overview and main results . . . . .     | 4           |
| 1.2 Preliminaries . . . . .                 | 5           |
| 1.3 Decision problems . . . . .             | 7           |
| <b>Chapter 2 Triangulations</b>             | <b>10</b>   |
| 2.1 Triangulation coordinates . . . . .     | 10          |
| 2.2 Nearby triangulations . . . . .         | 11          |
| 2.3 Existing algorithms . . . . .           | 12          |
| 2.4 Twist paths . . . . .                   | 13          |
| 2.4.1 Twisting a small curve . . . . .      | 14          |
| 2.4.2 Shortening a curve . . . . .          | 14          |
| 2.4.3 Half twisting a small curve . . . . . | 17          |
| 2.4.4 Isolating curves . . . . .            | 18          |
| <b>Chapter 3 Reducible mapping classes</b>  | <b>19</b>   |
| 3.1 Reducing curves . . . . .               | 19          |
| 3.2 Small vectors in polytopes . . . . .    | 20          |

|                  |   |           |
|------------------|---|-----------|
| 3.3              | Bounds on reducing curves . . . . .                   | 21        |
| 3.4              | Subsurfaces . . . . .                                 | 23        |
| 3.4.1            | Bounds on maximal curves . . . . .                    | 25        |
| 3.4.2            | The canonical curve system . . . . .                  | 27        |
| <b>Chapter 4</b> | <b>Pseudo-Anosov mapping classes</b>                  | <b>28</b> |
| 4.1              | Measured laminations . . . . .                        | 28        |
| 4.2              | Train tracks . . . . .                                | 31        |
| 4.2.1            | Maximal splittings and the axis . . . . .             | 33        |
| 4.2.2            | Getting to the axis . . . . .                         | 34        |
| 4.3              | The main algorithm . . . . .                          | 36        |
| 4.4              | Algebraic numbers . . . . .                           | 39        |
| 4.5              | Correctness . . . . .                                 | 39        |
| 4.5.1            | Acceptance implies pseudo-Anosov . . . . .            | 40        |
| 4.5.2            | Pseudo-Anosovs have acceptable certificates . . . . . | 43        |
| 4.6              | Analysis . . . . .                                    | 46        |
| <b>Chapter 5</b> | <b>The conjugacy problem</b>                          | <b>48</b> |
| 5.1              | Periodic mapping classes . . . . .                    | 49        |
| 5.2              | Periodic irreducible mapping classes . . . . .        | 54        |
| 5.3              | Periodic reducible mapping classes . . . . .          | 55        |
| 5.4              | Aperiodic irreducible mapping classes . . . . .       | 57        |
| 5.5              | Aperiodic reducible mapping classes . . . . .         | 58        |
| 5.5.1            | Twist invariants . . . . .                            | 59        |
| 5.5.2            | Equivalence of partition graphs . . . . .             | 60        |
| <b>Chapter 6</b> | <b>Other applications and implementation</b>          | <b>62</b> |
| 6.1              | Applications . . . . .                                | 62        |
| 6.1.1            | Nielsen–Thurston types . . . . .                      | 62        |
| 6.1.2            | Dilatation . . . . .                                  | 63        |
| 6.1.3            | Stratum . . . . .                                     | 63        |
| 6.1.4            | Orientability . . . . .                               | 64        |
| 6.1.5            | Commuting . . . . .                                   | 65        |
| 6.1.6            | Roots . . . . .                                       | 65        |
| 6.1.7            | Special subgroups . . . . .                           | 66        |
| 6.2              | Implementation . . . . .                              | 67        |
| 6.3              | Examples . . . . .                                    | 70        |
| 6.3.1            | On the once-marked torus . . . . .                    | 70        |

|                                       |   |           |
|---------------------------------------|---|-----------|
| 6.3.2                                 | On the twice-marked torus . . . . .                         | 71        |
| 6.3.3                                 | On a higher genus surface . . . . .                         | 72        |
| 6.3.4                                 | Penner like examples . . . . .                              | 74        |
| 6.4                                   | Further extensions . . . . .                                | 76        |
| 6.4.1                                 | The extended mapping class group . . . . .                  | 76        |
| 6.4.2                                 | Other surfaces . . . . .                                    | 76        |
| 6.4.3                                 | Independence of surface . . . . .                           | 77        |
| <b>Appendix A flipper timings</b>     |   | <b>80</b> |
| A.1                                   | On the once-marked torus . . . . .                          | 80        |
| A.2                                   | On the twice-marked torus . . . . .                         | 81        |
| A.3                                   | A comparison with Bestvina–Handel implementations . . . . . | 81        |
| <b>Appendix B flipper source code</b> |   | <b>83</b> |
| B.1                                   | Invariant laminations . . . . .                             | 83        |
| B.2                                   | Splitting sequence . . . . .                                | 86        |
| <b>Appendix C Censuses</b>            |   | <b>88</b> |
| C.1                                   | Monodromies of fibred knots . . . . .                       | 90        |
| C.2                                   | Monodromies of fibred census manifolds . . . . .            | 117       |

# List of Tables

|      |  |     |
|------|--|-----|
| 5.1  | Exceptional orbifolds which are almost triangle orbifolds. . . . . | 52  |
| 6.1  | A first Penner like family. . . . .                                | 75  |
| 6.2  | A second Penner like family. . . . .                               | 76  |
| C.1  | Fibred knot complements with fibre $S_{1,1}$ . . . . .             | 90  |
| C.2  | Fibred knot complements with fibre $S_{2,1}$ . . . . .             | 90  |
| C.3  | Fibred knot complements with fibre $S_{3,1}$ . . . . .             | 91  |
| C.4  | Fibred knot complements with fibre $S_{4,1}$ . . . . .             | 104 |
| C.5  | Fibred knot complements with fibre $S_{5,1}$ . . . . .             | 116 |
| C.6  | Fibred census manifolds with fibre $S_{1,1}$ . . . . .             | 117 |
| C.7  | Fibred census manifolds with fibre $S_{2,1}$ . . . . .             | 118 |
| C.8  | Fibred census manifolds with fibre $S_{3,1}$ . . . . .             | 122 |
| C.9  | Fibred census manifolds with fibre $S_{4,1}$ . . . . .             | 124 |
| C.10 | Fibred census manifolds with fibre $S_{5,1}$ . . . . .             | 125 |

# List of Figures

|     |   |    |
|-----|---|----|
| 1.1 | The figure 8 knot. . . . .  | 3  |
| 1.2 | A left Dehn twist about $\gamma$ . . . . .                                    | 7  |
| 1.3 | A left half twist about $\gamma$ . . . . .                                    | 7  |
| 1.4 | The Labruère–Paris generating set [55, Figure 13]. . . . .                    | 7  |
| 1.5 | Relationships between complexity classes. . . . .                             | 9  |
| 2.1 | Flipping an edge of a triangulation. . . . .                                  | 12 |
| 2.2 | A non-flippable edge $e$ . . . . .  | 14 |
| 2.3 | A Dehn twist along a small curve. . . . .                                     | 14 |
| 2.4 | The nearby points of $z$ . . . . .  | 15 |
| 2.5 | A curve meeting each edge at most once. . . . .                               | 16 |
| 2.6 | A half twist along a small curve. . . . .                                     | 17 |
| 2.7 | A chain on $S_{4,1}$ . . . . .  | 18 |
| 3.1 | Crushing along a multicurve. . . . .  | 24 |
| 3.2 | Building the dual graph of $\mathcal{T}_\gamma$ inside $S - \gamma$ . . . . . | 25 |
| 4.1 | The possibilities for splitting a branch. . . . .                             | 32 |
| 4.2 | Getting to the axis $A$ . . . . .   | 36 |
| 4.3 | A train track coming from a triangulation. . . . .                            | 37 |
| 5.1 | Lifting an arc from a triangle orbifold. . . . .                              | 54 |
| 5.2 | A partition graph. . . . .  | 58 |
| 6.1 | A singularity of order 4. . . . .   | 64 |
| 6.2 | The <b>flipper</b> GUI. . . . .   | 68 |
| 6.3 | Curves on $S_{1,1}$ . . . . .   | 70 |
| 6.4 | Curves on $S_{1,2}$ . . . . .   | 72 |
| 6.5 | The Whitehead link. . . . .   | 72 |
| 6.6 | Curves on $S_{2,1}$ . . . . .   | 73 |

|     |   |    |
|-----|---|----|
| 6.7 | Curves on $S_{4,2}$ .                       | 75 |
| 6.8 | An invariant lamination.                    | 77 |
| A.1 | <b>flipper</b> and Bestvina–Handel timings. | 82 |
| C.1 | Generators of $\text{Mod}^+(S)$ .           | 89 |



# Acknowledgments

I would not have been able to complete this work without the support of many people.

Firstly I would like to thank my supervisor Saul Schleimer, who has continuously provided good advice and guidance throughout my Ph.D. I also wish to convey my thanks to the members of the Warwick geometry and topology group for many stimulating discussions. In particular the current and past postgraduate students; Tom, Nick, Francesca, Damiano, Sara, Rupert, Rob, Katie, Richard and Alex, whom I have shared this experience with.

Secondly I thank all of my friends, both at Warwick and further afield. I would especially like to thank Florian, Jenny, Helene, Céline, Dave, Vandita, Ben, Dan, Alex, Ian and Chris who have endured far too many questions of mine.

Thirdly I am grateful to the Engineering and Physical Sciences Research Council for funding my studies and for the computational resources made available to me both from the University of Warwick IT Services and from John Cremona and the Warwick number theory group. Without which, much of this work would not have been possible. I also wish to extend my thanks to the entire administrative team at the Warwick Mathematics Institute and particularly Carole Fisher.

Finally, I wish to thank my family for all of their ongoing support. Most of all my darling wife Elizabeth who has been an inspiration throughout all of my studies.

This thesis was typeset with  $\text{\LaTeX} 2_{\epsilon}$ <sup>1</sup> by the author.

---

<sup>1</sup> $\text{\LaTeX} 2_{\epsilon}$  is an extension of  $\text{\LaTeX}$ .  $\text{\LaTeX}$  is a collection of macros for  $\text{\TeX}$ .  $\text{\TeX}$  is a trademark of the American Mathematical Society. The style package *warwickthesis* was used.

Beautiful is better than ugly.  
Explicit is better than implicit.  
Simple is better than complex.  
Complex is better than complicated.  
Flat is better than nested.  
Sparse is better than dense.  
Readability counts.  
Special cases aren't special enough to break the rules.  
Although practicality beats purity.  
Errors should never pass silently.  
Unless explicitly silenced.  
In the face of ambiguity, refuse the temptation to guess.  
There should be one – and preferably only one – obvious way to do it.  
Although that way may not be obvious at first unless you're Dutch.  
Now is better than never.  
Although never is often better than \*right\* now.  
If the implementation is hard to explain, it's a bad idea.  
If the implementation is easy to explain, it may be a good idea.  
Namespaces are one honking great idea – let's do more of those!

*The Zen of Python* by Tim Peters

# Declarations

I declare that the material in this thesis is, to the best of my knowledge, my own except where otherwise indicated or cited in the text, or else where the material is widely known. This material has not been submitted for any other degree.

Some of the material of Chapters 2 and 3 appears in an earlier preprint [8]. Additionally some of the material of Chapters 4 and 5 appears in a preprint [9] currently under submission.

All tables and figures were created by the author using Inkscape, Python, Sage and TikZ.

# Abstract

This thesis focuses on three decision problems in the mapping class groups of surfaces, namely the reducibility, pseudo-Anosov and conjugacy problems. For a fixed surface, we use ideal triangulations to model both its mapping class group and space of measured laminations. This allows us to state these problems combinatorially. We give new solutions to each of these problems that, unlike the existing solutions which are based on the Bestvina–Handel algorithm, run in polynomial time when given a suitable certificate. This allows us to show that in fact each of these problems lies in the complexity class  $\mathbf{NP} \cap \mathbf{co-NP}$  instead of just  $\mathbf{EXPTIME}$ .

At the heart of each of our solutions is the maximal splitting sequence of a projectively invariant measured lamination, as described by Agol. The complexity of this sequence bounds the difficulty of determining many of the properties of such a lamination, including whether it is filling. In Chapter 4 we give explicit polynomial upper bounds on the periodic and preperiodic lengths of such a sequence. This allows us to construct the running time bounds needed to show that these problems lie in  $\mathbf{NP} \cap \mathbf{co-NP}$ .

We finish with a discussion of an implementation of these algorithms as part of the Python package `flipper`. We include several examples of properties of mapping classes that can be computed using it.

# Abbreviations

Throughout we use:

- $S_{g,n}$  to denote the closed, connected, orientable surface of genus  $g$  with  $n$  marked points.
- $\text{ord}(g)$  to denote the order of a group element  $g$ .
- $\mathbb{N}$  to denote the set of natural numbers, including zero.
- $\mathbb{R}_{\geq 0}$  to denote the set of non-negative real numbers.
- $N(x)$  to denote a closed regular neighbourhood of  $x$ .
- $\text{Id}$  to denote the identity matrix, the dimension of which will always be evident from the context.
- $\begin{pmatrix} A \\ B \end{pmatrix}$  to denote the join of matrices  $A$  and  $B$  obtained by stacking their rows.

Furthermore, for a marked surface  $S$  we use:

- $g(S)$  to denote the genus of  $S$ ,
- $n(S)$  to denote the number of marked points of  $S$ , and
- $|S|$  to denote the number of connected components of  $S$

and for a vector  $v$  we use:

- $v \geq 0$  to denote that  $v$  is *non-negative*, that is, each entry of  $v$  is non-negative,
- $v \geq_2 0$  to denote that  $v \geq 0$  and that each entry of  $v$  is an even integer, and

- $\|v\|$  to denote the 1-norm of  $v$ .

# Chapter 1

## Introduction

Mapping class groups of surfaces have been studied for almost a century [33]. First introduced by Dehn, their natural actions on spaces including the curve complex,  $\mathcal{PML}$  and Teichmüller space were studied extensively by Nielsen, Teichmüller, Bers, Harvey and Thurston. Many of these spaces are combinatorial in nature and this has allowed problem regarding mapping classes to be tackled algorithmically.

A key result in the theory of mapping classes, started by Nielsen and completed by Thurston, is their classification:

**Theorem 1.0.1** (Nielsen–Thurston classification [37, Theorem 13.2]). *For any connected marked surface  $S$ , each mapping class is either:*

- periodic, *that is, finite order;*
- reducible, *that is, fixes a multicurve; or*
- pseudo-Anosov, *that is, projectively fixes a filling measured lamination.*

*Moreover, a mapping class is pseudo-Anosov if and only if it is neither periodic nor reducible.* □

This is perhaps easiest to see in the case of the once-marked torus. As a motivating example, consider the linear transformation of  $\mathbb{R}^2$  given by the matrix

$$M := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

This homeomorphism of the plane preserves the integer lattice  $\mathbb{Z}^2$  and so descends to a homeomorphism  $\phi$  of the once-marked torus

$$S_{1,1} := \mathbb{R}^2 / \mathbb{Z}^2,$$

where the marked point lifts to the lattice. There are several different ways to see that  $h = [\phi]$ , the mapping class represented by  $\phi$ , does not fix any simple closed curve. A straightforward way to see this is to note that if there were such a curve then it would lift back to an invariant line in  $\mathbb{R}^2$ . This line would have rational slope and  $M$  would have 1 as an eigenvalue, which it does not.

An alternative, and much more closely related to the techniques we will develop in Chapter 4, way to see this is to note that  $M$  has

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \varphi \\ 1 \end{pmatrix}$$

as an eigenvector, where  $\varphi = \frac{1+\sqrt{5}}{2}$  is the golden ratio. This gives rise to a lamination  $\mathcal{L}$  on  $S_{1,1}$  which is projectively invariant under  $h$ . Now any curve which meets  $\mathcal{L}$  cannot be invariant under  $h$  as the measure assigned to it by  $\mathcal{L}$  changes by a factor of  $\varphi > 1$  when  $h$  is applied. Thus we need only to discount the possibility of a loop disjoint from  $\mathcal{L}$ , running parallel to its leaves. To do this, we note that  $\varphi := 1 + \frac{1}{\varphi}$  and so the continued fraction expansion of  $v_1/v_2$  is

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\dots}}}$$

Therefore the line which is disjoint from  $\mathcal{L}$  has slope  $v_1/v_2$  which is irrational and so does not correspond to a simple closed curve. Hence  $h$  is irreducible.

Furthermore, as  $M, \dots, M^6 \neq \text{Id}$ , by Landau's estimate [57, Page 631]  $h$  is aperiodic and so we deduce that  $h$  is pseudo-Anosov.

In fact, every mapping class on the once-marked torus descends from such a linear transformation [37, Section 2.2.4]. The previous argument can be applied to any aperiodic mapping class. Its corresponding matrix in  $\text{SL}(2, \mathbb{Z})$  has a real eigenvector  $(v_1 \ v_2)^T$  giving a projectively invariant lamination  $\mathcal{L}$  on  $S_{1,1}$ . Assuming that  $v_2 \neq 0$ , the ratio  $v_1/v_2$  is an algebraic number and lies in a real quadratic extension. Thus its continued fraction expansion

$$v_1/v_2 = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots}}} = [a_0, a_1, a_2, \dots]$$

is either finite or eventually periodic [51, Theorem 28]. Again, these two cases differentiate between when the line with slope  $v_1/v_2$  corresponds to an invariant simple closed curve or not respectively. The eventual periodicity of this sequence



gives us a termination criteria and so a finite algorithm to decide which of these two cases we are in.

Although a similar result also holds for  $S_{0,4}$  and matrices in  $\mathrm{PSL}(2, \mathbb{Z})$  [37, Proposition 2.6], the mapping class group of a general surface is not even known to be linear [36, Page 16]. Despite this, we will show that a version of this construction can still be done.

Mapping classes also appear in low-dimensional topology in the form of fibred 3-manifolds. The *mapping torus* with *monodromy*  $\phi \in \mathrm{Homeo}^+(S)$  is the 3-manifold:

$$M_\phi := (S - P) \times [0, 1] / (x, 0) \sim (\phi(x), 1)$$

where  $P$  is the set of marked points of  $S$ . As the homeomorphism class of  $M_\phi$  is invariant under isotopy of  $\phi$ , we also denote this by  $M_h$  where  $h = [\phi]$ . For such manifolds their geometry is determined entirely by the Nielsen–Thurston type of their monodromy.

**Theorem 1.0.2** ([37, Theorem 13.4]). *The mapping torus  $M_h$  is hyperbolic if and only if  $h \in \mathrm{Mod}^+(S)$  is pseudo-Anosov.*  $\square$

This allows us to construct many examples of hyperbolic 3-manifolds using the previous procedure. The mapping class  $h \in \mathrm{Mod}^+(S_{1,1})$ , defined above by the matrix  $M$ , is perhaps the most famous of these. Its mapping torus  $M_h$  is homeomorphic to the complement of the figure 8 knot shown in Figure 1.1 [22, Section 5.14] [41, Chapter 8]. Thus, as  $h$  is pseudo-Anosov this is a hyperbolic knot [87, Page 4].

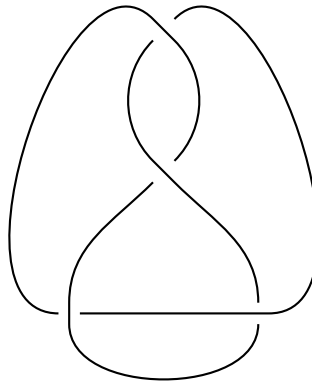


Figure 1.1: The figure 8 knot.

Mapping tori are also of considerable interest thanks to resolution of the virtual fibering conjecture:

**Theorem 1.0.3** (The virtual fibering conjecture [4, Theorem 9.2]). *Every hyperbolic 3-manifold has a finite index cover which is a mapping torus.*  $\square$

The surfaces  $S_{0,n}$  ( $n \geq 4$ ) have also been of particular interest and where significantly more progress has been made. Here determining reducibility for an element of  $\text{Mod}^+(S_{0,n})$  is closely related to determining the reducibility of a braid in  $B_{n-1}$ , the braid group on  $n - 1$  strands. This appears as Problem 6 of [16, Page 216]. Los first showed that this was decidable [58]. Using Garside normal form, Benardete, Gutierrez and Nitecki improved this to give an exponential time solution [12]. For  $B_4$ , Calvez and Wiest gave a solution which is quadratic in braid length [26]. However, their results are specific to  $B_4$  and do not generalise. Most recently Calvez showed that for each  $n$  there is an algorithm for determining reducibility for  $B_n$  whose runtime is cubic in braid length [25, Theorem 1]. However, this result uses the Linearly Bounded Conjugator Property of the mapping class group [84, Theorem B] [59, Theorem 7.2]. As this linear bound is not explicitly known, this is an existence result and is not constructive.

For the conjugacy problem, again the Garside structure of the braid group gives rise to an exponential time solution [43] [15].

## 1.1 Overview and main results

We fix a marked surface  $S$  and a finite generating set  $X$  of its mapping class group.

In Chapter 2 we introduce our main tool: ideal triangulations of  $S$ . We use these to combinatorially model both multicurves on  $S$  and the mapping class group of  $S$ . Furthermore we give an explicit description of how to construct a sequence of triangulations representing a Dehn twist and a half twist about a curve. This is based on a sequence of lemmas (Lemma 2.4.3 – Lemma 2.4.5) that allow us to simplify a triangulation with respect to a curve.

In Chapter 3 we introduce the weight space of a triangulation which we use to encode multicurves. We use the action of the mapping class group on this space to bound the complexity of reducing curves (Theorem 3.3.3). We use this to tackle the reducibility problem and show that it lies in **NP** (Corollary 3.3.5). We also introduce the canonical curve system of a mapping class, which was shown to be bounded exponentially by Koberda and Mangahas [54, Theorem 1.1]. By showing how to induct on subsurfaces and control the complexity of the transition, we give an alternate proof of their result (Corollary 3.4.11).

In Chapter 4 we generalise the weight space of a triangulation, along with our model of computation, to allow us to combinatorially represent and manipulate

measured laminations on  $S$ . Here we describe the main algorithm of this thesis (Section 4.3); an algorithm to determine whether a mapping class is pseudo-Anosov. Our approach relies on the maximal splitting move of a measured train track, originally described by Agol. By constructing an explicit upper bound on the number of these moves needed (Theorem 4.2.9 and Theorem 4.2.12) we show that deciding whether a mapping class is pseudo-Anosov is also a problem in **NP** (Corollary 4.6.1). The upper bound of Theorem 4.2.12 relies on the Linearly Bounded Conjugator Property of the mapping class group [84, Theorem B] [59, Theorem 7.2]. This means that the main algorithm suffers from the same problems as Calvez’s algorithm for deciding reducibility in  $B_n$  [25, Theorem 1]. We describe a minor change that can be made to the main algorithm so that knowledge of the linear bounding constant is not needed in order to run it (Remark 4.3.2).

In Chapter 5 we use these tools to tackle the conjugacy problem for mapping class groups. We do this by breaking into four cases: depending on whether the mapping class is periodic or aperiodic and is reducible or irreducible. For each we describe a total conjugacy invariant:

- In the periodic case we use the quotient orbifold.
- In the aperiodic, irreducible case we use the periodic part of the maximal splitting sequence.
- In the aperiodic, reducible case we use its partition graph.

Again we construct a bound on the complexity of the construction of each of these invariants. This extends a result of Tao [84, Theorem B] to show that the conjugacy problem lies in **NP**  $\cap$  **co-NP** (Corollary 5.5.6).

In Chapter 6 we describe several additional properties of mapping classes and show that for each the problem of computing them lies in **NP** (or its equivalent functional class **FNP**). We give details of an implementation of the main algorithm, as part of the Python package `flipper`, along with several examples of the kinds of calculations that can be done using it. Further information about `flipper` can also be found in Appendix A and Appendix B. We finish with some further extensions and conjectures, in particular the difficulties that arise when  $S$  is allowed to vary.

## 1.2 Preliminaries

We now set some additional notation and terminology.

Let  $S$  be a marked surface and  $P$  its set of marked points. We allow  $S$  to have finitely many connected components, each of finite type. However, due to a technical

restriction which appears in Chapter 2, we require that none of these components are  $S_{g,0}$ ,  $S_{0,1}$  or  $S_{0,2}$ . The *mapping class group* of  $S$  is

$$\text{Mod}^+(S) := \text{Homeo}^+(S) / \text{isotopy}$$

where  $\text{Homeo}^+(S)$  is the group of orientation-preserving homeomorphisms of  $S$  and the isotopies are performed relative to  $P$  [37, Chapter 2]. For a detailed discussion of mapping class groups we refer the reader to [37] and [39].

The mapping class group of  $S$  is finitely generated [37, Theorem 5.7] and we fix  $X$  to be one such finite generating set. We consider the word  $h = h_1 \cdots h_k \in X^*$  as representing the mapping class  $h_1 \circ \cdots \circ h_k$  and write  $\ell(h) := k$  for the *length* of  $h$ . Additionally, we write  $g \equiv h$  if two words  $g, h \in X^*$  represent the same mapping class.

A *multicurve*  $\gamma$  on  $S$  is the isotopy class of the image of a smooth embedding of a finite number of copies of  $S^1$  into  $S$ , disjoint from  $P$ , such that no component of  $S - \gamma$  is a disk or once-marked disk. We denote the set of all multicurves on  $S$  by  $\mathcal{C}(S)$ . This is a strictly larger set than the set of simplices of the curve complex of  $S$  [59]; it includes multicurves in which some of the components are parallel. A multicurve is a *curve* if it has exactly one component. To avoid a recurring edge case we will disallow the *empty multicurve*, the multicurve with zero components, throughout.

We use curves on  $S$  to describe two types of elementary mapping classes.

### Dehn twists

Suppose that  $\gamma \in \mathcal{C}(S)$  is a curve. The *left Dehn twist* about  $\gamma$ , denoted  $T_\gamma$ , is the mapping class that acts on a neighbourhood of  $\gamma$  as shown in Figure 1.2 and by the identity elsewhere [37, Section 3.1.1]. Its inverse,  $T_\gamma^{-1}$ , is also referred to as a *right Dehn twist*. If  $\gamma$  is a multicurve then we use  $T_\gamma$  to denote the composition of Dehn twists about each of its components. These commute and so the order of composition is not important.

### Half twists

A curve  $\gamma \in \mathcal{C}(S)$  is *half-twistable* if a component of  $S - \gamma$  is a twice-marked disk. For such a curve there is a topological square root of  $T_\gamma$ , the *half twist* about  $\gamma$ , denoted  $T_\gamma^{1/2}$ . This acts on a neighbourhood of the twice-marked disk as shown in Figure 1.3 and by the identity elsewhere [37, Section 9.1.3].

When  $S$  is connected, Dehn twists and half twists generate  $\text{Mod}^+(S)$ . In fact  $\text{PMod}^+(S)$ , the subgroup consisting of the elements of  $\text{Mod}^+(S)$  that fix each

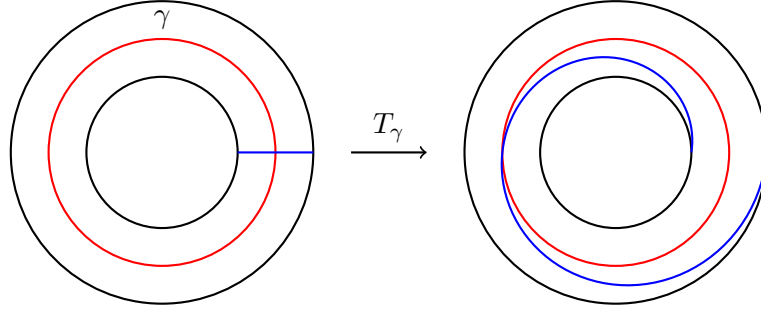


Figure 1.2: A left Dehn twist about  $\gamma$ .

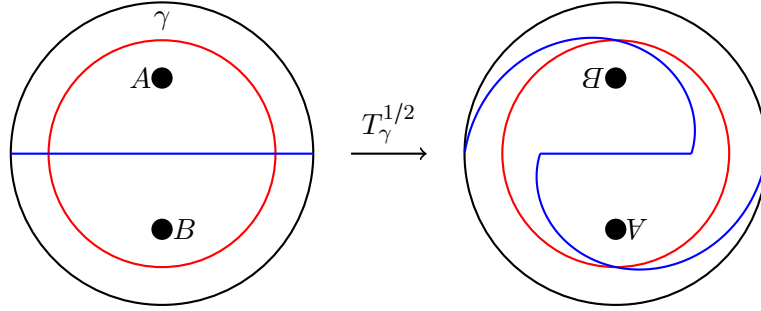


Figure 1.3: A left half twist about  $\gamma$ .

marked point, is generated by Dehn twists along the  $2g + n + 1$  non-separating curves shown in Figure 1.4. These twists along with  $n - 1$  half twists that exchange each pair of marked points generate  $\text{Mod}^+(S)$  [55, Proposition 2.10].

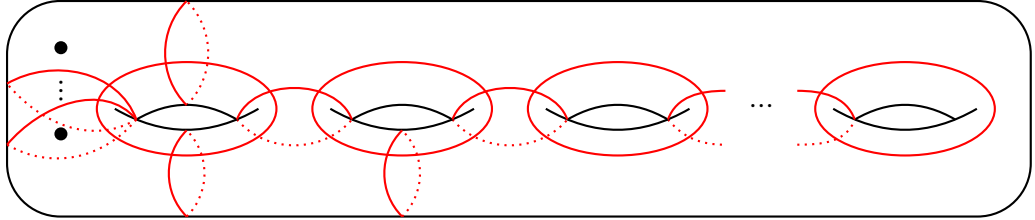


Figure 1.4: The Labruère–Paris generating set [55, Figure 13].

### 1.3 Decision problems

We also recall some standard definitions from computational complexity theory. We will be concerned with *decision problems*, a parametrised family of questions each with a yes / no answer. A decision problem  $Q$  is *decidable* if there is a Turing machine  $M$  such that for each *instance*  $\omega \in Q$ :  $M$  accepts  $\omega$  if  $Q(\omega)$  is true and rejects it

otherwise [82, Chapter 3]. Additionally, for an instance  $\omega \in Q$  we denote the size of the input to  $M$  by  $|\omega|$ .

The following complexity classes will appear throughout this thesis.

**Definition 1.3.1** ([48, Appendix A.1]). A decision problem  $Q$  is in **P** if there is a polynomial  $q \in \mathbb{Z}[x]$  and Turing machine  $M$  such that for each instance  $\omega \in Q$ :  $M$  accepts  $\omega$  in time  $q(|\omega|)$  if and only if  $Q(\omega)$  is true.

These are the problems that can be efficiently solved by a computer.

**Definition 1.3.2** ([48, Appendix A.2]). A decision problem  $Q$  is in **NP** if there is a polynomial  $q \in \mathbb{Z}[x]$  and Turing machine  $M$  such that for each instance  $\omega \in Q$ : there is a *certificate*  $c$  such that  $M$  accepts  $(\omega, c)$  in time  $q(|\omega|)$  if and only if  $Q(\omega)$  is true.

These are the problems for which if an instance is true then there is a small proof of this which can be efficiently verified by a computer.

**Definition 1.3.3** ([48, Appendix A.4]). A decision problem  $Q$  is in **co-NP** if there is a polynomial  $q \in \mathbb{Z}[x]$  and Turing machine  $M$  such that for each instance  $\omega \in Q$ : there is a certificate  $c$  such that  $M$  accepts  $(\omega, c)$  in time  $q(|\omega|)$  if and only if  $Q(\omega)$  is false.

These are the problems for which if an instance is false then there is a small counterexample which can be verified efficiently.

It is well known that **NP**  $\subseteq$  **EXPTIME**, the class of problems solvable in exponential time [48, Appendix A.5]. Moreover, given a polynomial-time algorithm for validating certificates of a problem  $Q \in \mathbf{NP}$  there is a method for building an exponential time algorithm for solving  $Q$ . There is an analogous construction for **co-NP**  $\subseteq$  **EXPTIME**. Other relations, such as **P** = **NP**, **NP** = **co-NP** and **P** = **NP**  $\cap$  **co-NP**, remain major open problems in computational complexity theory.

We will begin by working only with integers and, to simplify our analysis, we will assume that in our model of computation there is no cost associated to accessing variables. For a function  $f: \mathbb{N}^k \rightarrow \mathbb{N}$  we use the standard computational complexity notation of:

$$O(f) := \{g: \mathbb{N}^k \rightarrow \mathbb{N} : \exists C \text{ such that } g(x) \leq Cf(x) \text{ for all but finitely many } x\}.$$

As expected if  $f' \in O(f)$  and  $g' \in O(g)$  then

$$f' + g' \in O(f + g) \quad \text{and} \quad f'g' \in O(fg).$$

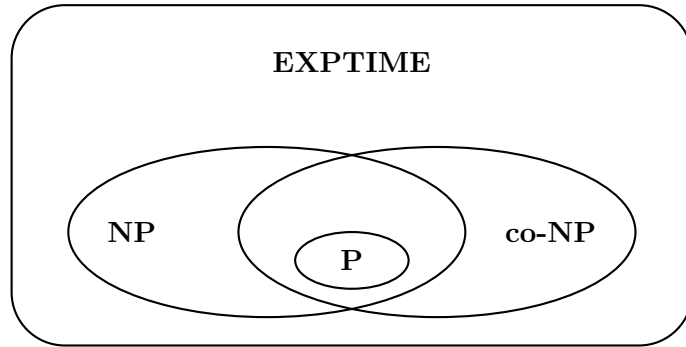


Figure 1.5: Relationships between complexity classes. Note that many of these containments are not known to be strict.

**Definition 1.3.4.** An integer  $x$  is  $k$ -bounded if it has at most  $k$  digits, that is, if it can be represented by  $O(k)$  bits.

As part of our model of computation, we will assume that if  $x$  and  $y$  are  $k$ -bounded then:

- $\text{sign}(x)$  can be computed in  $O(1)$  operations,
- $x \pm y$  is  $(k + 1)$ -bounded and can be computed in  $O(k)$  operations, and
- $xy$  is  $2k$ -bounded and can be computed in  $O(k^2)$  operations.

More generally, we will say that a vector (or matrix) is  $k$ -bounded if each of its entries is  $k$ -bounded. Thus a  $k$ -bounded,  $m \times n$  matrix can be represented by  $O(mnk)$  bits.

## Chapter 2

# Triangulations

The purpose of this chapter is to build a combinatorial model of  $\text{Mod}^+(S)$  using ideal triangulations. These triangulations will also allow us to describe curves on  $S$ . We go on to give a method for constructing explicit representations of a Dehn twist and half twist about certain curves, which includes a generating set of  $\text{Mod}^+(S)$ .

### 2.1 Triangulation coordinates

**Definition 2.1.1.** An *(ideal) triangulation*  $\mathcal{T}$  of  $S$  is the isotopy class of the image of an embedding of a finite number of copies of  $[0, 1]$  into  $S$  such that:

- the endpoints of each interval are sent to marked points,
- no component is isotopic to a marked point, and
- the metric completion of each component of  $S - \mathcal{T}$  is an unmarked triangle

together with an ordering of the components.

As  $S$  cannot be triangulated when it has an  $S_{g,0}$ ,  $S_{0,1}$  or  $S_{0,2}$  component [79], we disallow these surfaces throughout.

To follow standard conventions, we refer to the marked points as *vertices*, the images of the intervals as *edges* and the metric completion of the components of  $S - \mathcal{T}$  as *faces*. We denote the sets of these by  $V(\mathcal{T})$ ,  $E(\mathcal{T})$  and  $F(\mathcal{T})$  respectively. We let

$$\zeta = \zeta(S) := 6g(S) + 3n(S) - 6|S| \geq 3|S|$$

denote the *complexity* of  $S$ . This is equal to the number of edges of any triangulation of  $S$ .



We think of the triangulation  $\mathcal{T}$ , with edges  $e_1, \dots, e_\zeta$ , as providing a coordinate system on  $\mathcal{C}(S)$  [62, Theorem 3.2.4]. The multicurve  $\gamma \in \mathcal{C}(S)$  corresponds to its *normal coordinate*, the point:

$$\mathcal{T}(\gamma) := \begin{pmatrix} \iota(\gamma, e_1) \\ \vdots \\ \iota(\gamma, e_\zeta) \end{pmatrix} \in \mathbb{N}^\zeta$$

where  $\iota(\alpha, \beta)$  denotes the (*geometric*) *intersection number* between  $\alpha$  and  $\beta$ . Although a multicurve is uniquely determined by  $\mathcal{T}(\gamma)$ , and so distinct multicurves have distinct coordinates, not all coordinates correspond to multicurves. In fact  $(v_1 \dots v_\zeta)^T$  corresponds to a multicurve if and only if:

- $\sum v_i > 0$ ,
- for each triangle  $a, b, c$  we have that  $v_a + v_b - v_c \geq 0$ , and
- for each vertex  $v$  there is a face with sides  $a, b, c$  such that  $v \subseteq a \cap b$  and  $v_a + v_b = v_c$ .

**Lemma 2.1.2.** *If  $v \in \mathbb{N}^\zeta$  is a  $k$ -bounded vector then we can determine whether  $v$  corresponds to a multicurve in  $O(k)$  operations.*  $\square$

## 2.2 Nearby triangulations

We use two elementary methods of altering a triangulation:

1. We may reorder the edges of  $\mathcal{T}$  using a permutation.
2. If an edge  $e \in E(\mathcal{T})$  meets two distinct faces then we may *flip* it. This is done by replacing  $e$  with the opposite diagonal of the square it is contained in, as shown in Figure 2.1.

Following this second move, we say that an edge  $e$  of  $\mathcal{T}$  is *flippable* if it meets two distinct faces. This move is also known as a 2–2 Pachner move [71, Definition 2.1].

We call a sequence of edge flips and reorderings, starting with the triangulation  $\mathcal{T}$  and finishing with  $\mathcal{T}'$ , as a *path* from  $\mathcal{T}$  to  $\mathcal{T}'$ . For such a path  $p$  we write  $\ell(p)$  for the number of edge flips that occur. Hatcher showed that any two triangulations are connected by such a path [47, Page 190]. This gives a pseudo-metric on the set of triangulations and relative to this the mapping class group acts on this space *geometrically*, that is, properly discontinuously, cocompactly and by isometries.

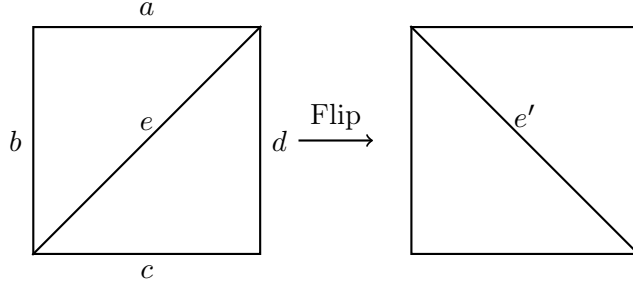


Figure 2.1: Flipping an edge of a triangulation.

Therefore there is an integer  $Q$  such that this space is  $Q$ -quasi-isometric to  $\text{Mod}^+(S)$  (with respect to  $X$ ) [19, Proposition 8.19].

To ease the notation in the following chapters, we fix a triangulation  $\mathcal{T}_*$  of  $S$ . Additionally, for each generator  $h \in X$  we fix a path from  $\mathcal{T}_*$  to  $h(\mathcal{T}_*)$  of length at most  $Q$ . Then for any word  $h \in X^*$  we obtain a path from  $\mathcal{T}_*$  to  $h(\mathcal{T}_*)$  of length at most  $Q\ell(h) \in O(\ell(h))$  by concatenating translations of these paths. We refer to this as the *standard path of  $h$* . Finally, we will simply say that a multicurve  $\gamma \in \mathcal{C}(S)$  is  $k$ -bounded if the vector  $\mathcal{T}_*(\gamma)$  is.

While it is obvious how the edge intersection numbers of a multicurve change under a reordering, the following proposition controls how they change under a flip.

**Proposition 2.2.1.** *Suppose that  $\gamma$  is a multicurve and  $e$  is a flippable edge of a triangulation  $\mathcal{T}$  as shown in Figure 2.1 then*

$$\iota(\gamma, e') = \max(\iota(\gamma, a) + \iota(\gamma, c), \iota(\gamma, b) + \iota(\gamma, d)) - \iota(\gamma, e). \quad \square$$

By tracking how the intersection numbers between a multicurve and the edges of a triangulation change under a sequence of flips and reorderings we obtain:

**Lemma 2.2.2.** *Suppose that  $h \in \text{Mod}^+(S)$  is a mapping class and  $p$  is a path from  $\mathcal{T}$  to  $h(\mathcal{T})$ . If  $\gamma \in \mathcal{C}(S)$  is a multicurve and  $\mathcal{T}(\gamma)$  is  $k$ -bounded then  $\mathcal{T}(h(\gamma))$  is  $(k + 3\ell(p))$ -bounded and can be computed in  $O(k\ell(p) + \ell(p)^2)$  operations.  $\square$*

## 2.3 Existing algorithms

This type of coordinate system on  $\mathcal{C}(S)$  has already been studied by various groups including Erickson and Nayyeri [35]; Agol, Hass and Thurston [5]; and Schaefer, Sedgwick and Štefankovič [80]. They have given several algorithms of which we highlight three that will be particularly useful over the coming chapters.

Suppose that  $\mathcal{T}$  is a triangulation and that  $\gamma$  and  $\gamma'$  are multicurves. The following algorithms take  $\mathcal{T}(\gamma)$  and  $\mathcal{T}(\gamma')$  as input, which we assume to be  $k$ -bounded. Firstly, we may extract the coordinate vectors and multiplicities of each of the components of  $\gamma$ :

**Proposition 2.3.1** ([35, Section 6] [5] [80, Theorem 1]). *There is an algorithm to compute  $\mathcal{T}(\gamma_0)$ , for each curve  $\gamma_0 \subseteq \gamma$ , along with the number of components in that class in  $O(\text{poly}(k))$  operations.*  $\square$

Using this one can also determine the total number of components of  $\gamma$  in  $O(\text{poly}(k))$  operations. Secondly, we can compute the image of one multicurve under the Dehn twist about another:

**Proposition 2.3.2** ([81, Theorem 4.1]). *Let  $\gamma'' := T_\gamma(\gamma')$  then  $\mathcal{T}(\gamma'')$  is  $2k$ -bounded and there is an algorithm to compute it in  $O(\text{poly}(k))$  operations.*  $\square$

From this one obtains an algorithm to compute the intersection number of two multicurves:

**Theorem 2.3.3** ([81, Lemma 5.4]). *The intersection number  $\iota(\gamma, \gamma')$  is  $2k$ -bounded and there is an algorithm to compute it in  $O(\text{poly}(k))$  operations.*  $\square$

## 2.4 Twist paths

We now describe a procedure for finding a path from  $\mathcal{T}$  to  $T_\gamma(\mathcal{T})$  whenever  $\gamma$  is a sufficiently nice curve. To algorithmically construct these we first require two technical lemmas.

**Lemma 2.4.1.** *Suppose that  $\mathcal{T}$  is a triangulation with edges  $e_1, \dots, e_\zeta$  then*

$$\iota(\alpha, \mathcal{T}) = \sum \iota(\alpha, e_i).$$

*Proof.* This result follows immediately from the *bigon criterion*: if  $a \in \alpha$  and  $b \in \beta$  are representatives then  $|a \cap b| = \iota(\alpha, \beta)$  if and only if  $a$  and  $b$  intersect transversely and do not form any *bigons*. [37, Proposition 1.7]  $\square$

**Lemma 2.4.2.** *If  $e$  maximises  $\iota(\gamma, \cdot)$  over all edges of  $\mathcal{T}$  then  $e$  is flippable.*

*Proof.* If  $e$  is not flippable then there is a face with sides  $a$ ,  $e$  and  $e$  as shown in Figure 2.2. Therefore  $\iota(\gamma, a) = 2\iota(\gamma, e)$  and so, as  $e$  maximises  $\iota(\gamma, \cdot)$  over all edges of  $\mathcal{T}$ , we have that  $\iota(\gamma, e) = 0$ . This contradicts the fact that  $\gamma$  is a curve.  $\square$

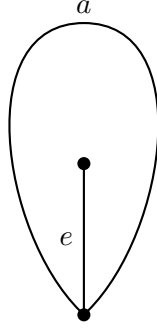


Figure 2.2: A non-flippable edge  $e$ .

### 2.4.1 Twisting a small curve

We start with a particularly simple case. Suppose that  $\gamma$  is a curve which meets  $\mathcal{T}$  as shown in Figure 2.3. This situation is characterised by the fact that  $\iota(\gamma, \mathcal{T}) = 2$ .

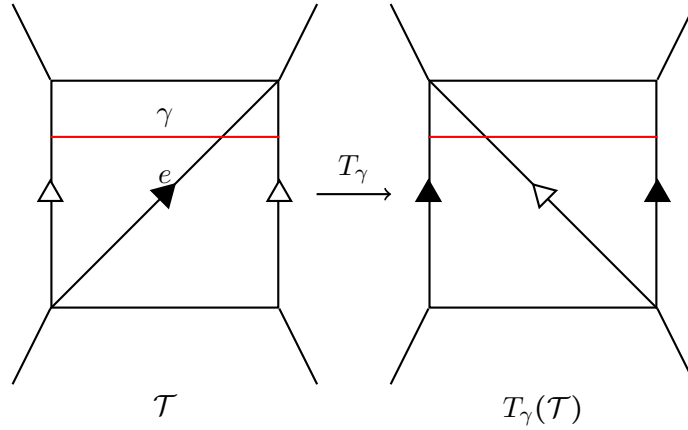


Figure 2.3: A Dehn twist along a small curve.

In this case we can give an explicit path from  $\mathcal{T}$  to  $T_\gamma(\mathcal{T})$  as they differ only by a flip of edge  $e$  and a reordering of edges.

### 2.4.2 Shortening a curve

For a more general curve  $\gamma$ , we aim to produce a sequence of flips to reduce  $\gamma$  to the case in Section 2.4.1. We say that an edge  $e$  of a triangulation  $\mathcal{T}$  is *shortening* (with respect to  $\gamma$ ) if  $e$  is flippable and  $\iota(\gamma, \mathcal{T}') < \iota(\gamma, \mathcal{T})$ , where  $\mathcal{T}'$  is the triangulation obtained by flipping  $e$ .

**Lemma 2.4.3.** *Let  $\mathcal{T}$  be a triangulation and  $\gamma$  a curve. If  $\iota(\gamma, e) > 2$  for some edge  $e$  of  $\mathcal{T}$  then there is a shortening edge.*

*Proof.* We will show the contrapositive, so suppose that every edge of  $\mathcal{T}$  is non-shortening. Now, abusing notation slightly, isotope  $\gamma$  so that it realises  $\iota(\gamma, \mathcal{T})$ . Let  $Z := \gamma \cap \mathcal{T}$  and for  $z \in Z$  define

$$\|z\| := \min(|\gamma \cap e_1|, |\gamma \cap e_2|)$$

where  $e_1$  and  $e_2$  are the two components of  $e - \{z\}$  and  $e$  is the edge of  $\mathcal{T}$  containing  $z$ . Additionally let  $k := \max_{e \in E(\mathcal{T})}(\iota(\gamma, e))$  and  $k' := \max_{z \in Z}(\|z\|)$ .

There is a point  $z \in Z$  such that  $\|z\| = k'$  and if  $e$  is the edge containing  $z$  then  $\iota(\gamma, e) = k$ . By Lemma 2.4.2,  $e$  must be flippable. Therefore, following the notation of Figure 2.1, as this is not a shortening edge we have that

$$\max(\iota(\gamma, a) + \iota(\gamma, c), \iota(\gamma, b) + \iota(\gamma, d)) - \iota(\gamma, e) \geq \iota(\gamma, e).$$

Hence it follows that either  $\iota(\gamma, a) = \iota(\gamma, c) = k$  or  $\iota(\gamma, b) = \iota(\gamma, d) = k$ . Without loss of generality we will assume the latter.

Now  $z$  is connected via  $\gamma$  to a point  $z' \in Z \cap b$  and  $z'' \in Z \cap d$ . From the previous calculation we deduce that  $\|z'\| = \|z''\| = k'$ . Thus by repeating this argument we see that, as  $\gamma$  is connected,  $\|\cdot\|$  is constant and so must in fact be the zero function. This shows that  $\iota(\gamma, e) \leq 2$  for each edge  $e$  and so the result holds.  $\square$

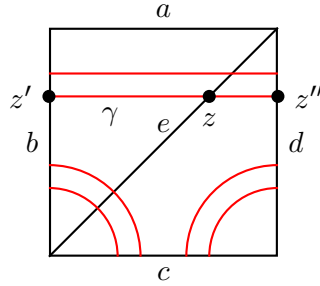


Figure 2.4: The nearby points of  $z$ . Here  $\|z\| = 1$ .

**Lemma 2.4.4.** *Let  $\mathcal{T}$  be a triangulation and  $\gamma$  a curve. If  $\iota(\gamma, \mathcal{T}) > 2$  and  $\iota(\gamma, e) \leq 1$  for every edge  $e$  then either:*

- *there is a shortening edge, or*
- *flipping any edge meeting  $\gamma$  preserves  $\iota(\gamma, \mathcal{T})$  and afterwards there is a shortening edge.*

*Proof.* Suppose that there are no shortening edges. Let  $e$  be an edge meeting  $\gamma$  then  $e$  must be flippable and, following the notation of Figure 2.1, without loss of generality we have that

$$\iota(\gamma, b) = \iota(\gamma, d) = 1 \quad \text{and} \quad \iota(\gamma, a) = \iota(\gamma, c) = 0.$$

Additionally,  $d$  must be flippable and  $\gamma$  must extend through the other triangle meeting  $d$  as shown in Figure 2.5 as otherwise  $d$  would be a shortening edge. In this case flipping  $e$  preserves  $\iota(\gamma, \mathcal{T})$  and afterwards  $d$  is a shortening edge.  $\square$

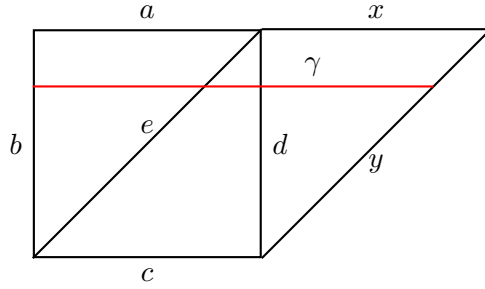


Figure 2.5: A curve meeting each edge at most once.

Unfortunately, we cannot bridge the conclusion of Lemma 2.4.3 to the hypothesis of Lemma 2.4.4 as this would imply that each component of  $S - \gamma$  contains at least one marked point. However, this is the only obstruction. Following this we say a curve  $\gamma$  is *isolating* if a component of  $S - \gamma$  contains no marked points. In particular, all non-separating curves are non-isolating.

**Lemma 2.4.5.** *Let  $\mathcal{T}$  be a triangulation and  $\gamma$  a non-isolating curve. If  $\iota(\gamma, e) > 1$  for some edge  $e$  then there is a shortening edge.*

*Proof.* Following Lemma 2.4.3, we may assume that  $\iota(\gamma, e) \leq 2$  for every edge  $e$ .

As  $\gamma$  is non-isolating, a parity argument shows that there is an edge which meets  $\gamma$  once. Hence there is a flippable edge  $e$  such that  $\iota(\gamma, e) = 2$  and, again following the notation of Figure 2.1, that

$$\iota(\gamma, a) = \iota(\gamma, b) = 1.$$

In this case by Proposition 2.2.1

$$\iota(\gamma, e') = \max(\iota(\gamma, c), \iota(\gamma, d)) - 1 < 2 = \iota(\gamma, e)$$

and so  $e$  is a shortening edge of  $\mathcal{T}$ .  $\square$

**Theorem 2.4.6.** *If  $\mathcal{T}$  is a triangulation and  $\gamma$  is a non-isolating curve then we can compute a path from  $\mathcal{T}$  to  $T_\gamma(\mathcal{T})$  of length at most  $4\iota(\gamma, \mathcal{T})$  in  $O(\iota(\gamma, \mathcal{T}))$  operations.*

*Proof.* By repeatedly applying Lemma 2.4.3, Lemma 2.4.5 and then Lemma 2.4.4 we obtain a path  $p_1$  from  $\mathcal{T}$  to a triangulation  $\mathcal{T}'$  such that  $\iota(\gamma, \mathcal{T}') = 2$ . Then, following Section 2.4.1, we can construct a path  $p_2$  from  $\mathcal{T}'$  to  $T_\gamma(\mathcal{T}')$ . Finally, the sequence of flips corresponding to the reverse of  $p_1$  applied to  $T_\gamma(\mathcal{T}')$  gives a path  $p_3$  from  $T_\gamma(\mathcal{T}')$  to  $T_\gamma(\mathcal{T})$ . Concatenating  $p_1$ ,  $p_2$  and  $p_3$  gives the required path.  $\square$

**Remark 2.4.7.** In the proof of each of Lemma 2.4.3, Lemma 2.4.4 and Lemma 2.4.5, the shortening edge found maximises  $\iota(\gamma, \cdot)$  over all edges of  $\mathcal{T}$ . Thus when performing this procedure one may keep the edges of  $\mathcal{T}$  in a priority queue [30, Section 6.5], ordered by intersection number with  $\gamma$ . Typically this significantly reduces the overhead of repeatedly finding a shortening edge.

### 2.4.3 Half twisting a small curve

Analogous to Section 2.4.1, suppose that  $\gamma$  is a curve which meets  $\mathcal{T}$  as shown in Figure 2.6. In this case one of the components of  $S - \gamma$  is a twice-marked disk and so we may perform a half twist about  $\gamma$ .

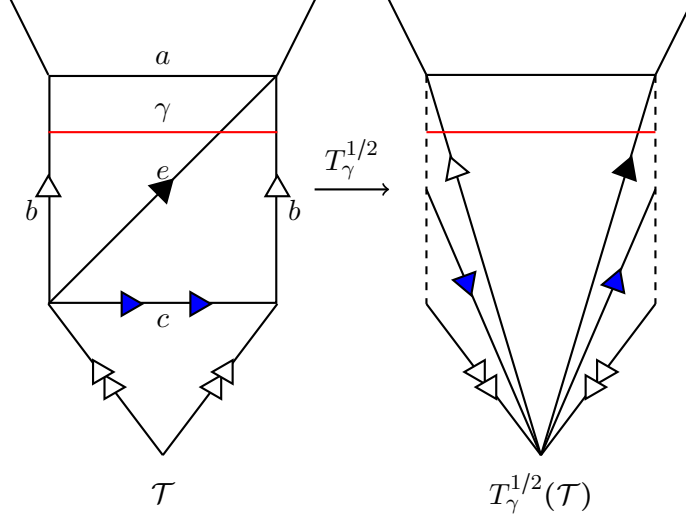


Figure 2.6: A half twist along a small curve.

Again, in this case we can give an explicit path from  $\mathcal{T}$  to  $T_\gamma^{1/2}(\mathcal{T})$ : flip edge  $c$ , then  $e$ , then  $b$  and reorder the edges. Again, the techniques described in Section 2.4.2 allow us to shorten any non-isolating curve bounding a twice-marked disk until it is as shown in Figure 2.6 and so we obtain:

**Theorem 2.4.8.** *If  $\mathcal{T}$  is a triangulation and  $\gamma$  is a non-isolating, half-twistable curve then we can compute a path from  $\mathcal{T}$  to  $T_\gamma^{1/2}(\mathcal{T})$  of length at most  $4\iota(\gamma, \mathcal{T})$  in  $O(\iota(\gamma, \mathcal{T}))$  operations.*  $\square$

#### 2.4.4 Isolating curves

Finally, in the case when  $\gamma$  is an isolating curve, we note that we can still apply the arguments of Section 2.4.2 to reduce to the case that  $\iota(\gamma, e) \in \{0, 2\}$  for every edge  $e$ . Let  $S'$  be the component of  $S - \gamma$  which does not contain any marked points. We find non-separating curves  $\gamma_1, \dots, \gamma_{2g'}$  (where  $g' := g(S')$ ) such that

$$\iota(\gamma_i, \gamma_j) = \begin{cases} 1 & \text{if } i = j \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

and  $N(\gamma_1 \cup \dots \cup \gamma_{2g'}) = S'$ . An example of such a chain when  $g(S') = 4$  is given in Figure 2.7. Then by the *chain relation* [37, Proposition 4.12]:

$$T_\gamma = (T_{\gamma_1} \circ \dots \circ T_{\gamma_{2g'}})^{4g'+2}.$$

Hence we can still compute a path from  $\mathcal{T}$  to  $T_\gamma(\mathcal{T})$ .

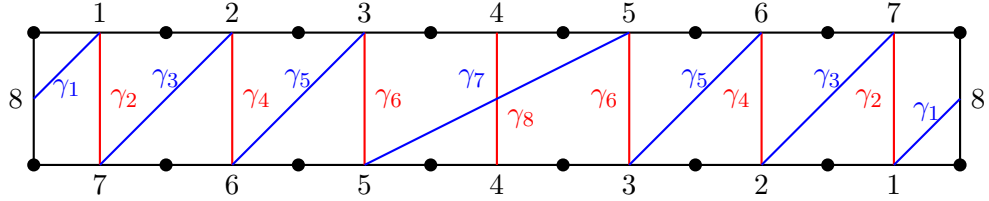


Figure 2.7: A chain on  $S_{4,1}$ . Sides with the same label are identified via a translation.

To perform a half-twist about an isolating curve  $\gamma$ , we first note that  $S$  must have exactly two marked points. As before we build a chain of length  $2g'$  in  $S'$  and combine

$$(T_{\gamma_1} \circ \dots \circ T_{\gamma_{2g'}})^{2g'+1}$$

with an involution of  $\mathcal{T}$  which interchanges the two marked points to obtain a path from  $\mathcal{T}$  to  $T_\gamma^{1/2}(\mathcal{T})$ .



## Chapter 3

# Reducible mapping classes

In this chapter we use the action of the mapping class group on the weight space of a triangulation to bound the number of bits needed to describe a reducing curve. We use this to tackle the reducibility problem and show that it lies in **NP**. Furthermore, by showing how to induct on subsurfaces and control the complexity of the transition we also give an alternate proof of a result of Koberda and Mangahas on the complexity of the canonical curve system of a mapping class [54, Theorem 1.1].

Much of the material of this chapter appears as part of [8].

### 3.1 Reducing curves

If  $h \in \text{Mod}^+(S)$  is a mapping class then we say that a multicurve  $\gamma \in \mathcal{C}(S)$  is *h-invariant* if  $h(\gamma) = \gamma$ .

**Definition 3.1.1** ([37, Section 13.2.2]). A mapping class  $h \in \text{Mod}^+(S)$  is *reducible* if there is an  $h$ -invariant multicurve. A word in  $X^*$  is *reducible* if its corresponding mapping class is.

**Problem 3.1.2** (The reducibility problem). Fix  $S$ , a marked surface, and  $X$ , a finite generating set of  $\text{Mod}^+(S)$ . Given a word  $h \in X^*$  decide whether  $h$  is reducible.

There are several algorithms to determine whether a mapping class is reducible. For example, this can be done using an adaptation of the Bestvina–Handel algorithm [13], which was originally used for the analogous problem for outer automorphisms of a free group using the theory of train-tracks [14]. However, the complexity of this algorithm is still unknown.

We use the piecewise-linear structure given to  $\mathcal{C}(S)$  by  $\mathcal{T}$  to show that if  $h \in X^*$  is reducible then it fixes a small multicurve. Such an invariant multicurve acts

as a certificate of reducibility and, as it is sufficiently small, that it can be verified in quadratic time. As usual, this gives another exponential time algorithm to determine whether a word is reducible.

### 3.2 Small vectors in polytopes

We will express the reducibility problem as a linear programming (LP) problem. Small invariant curves will then correspond to small solutions to this LP problem. This is closely related to the vertex enumeration problem for unbounded polytopes [18]. We start with a technical lemma for bounding determinants of matrices.

**Lemma 3.2.1.** *If  $M$  is a  $k$ -bounded,  $n \times n$  matrix then  $\det(M)$  is  $(kn + n \log(n)/2)$ -bounded.*  $\square$

*Proof.* This bound follows immediately from Hadamard's inequality [42, Theorem 14.1.1].  $\square$

**Proposition 3.2.2.** *Suppose that  $M$  is a  $k$ -bounded,  $m \times n$  matrix. If the polytope*

$$\mathbb{P} := \{v \in \mathbb{N}^n : M \cdot v \geq 0\}$$

*is non-trivial then it contains a non-trivial  $(nk + n \log(n)/2)$ -bounded integral vector.*

*Proof.* Without loss of generality we may assume that the basis vectors  $(0 \cdots 0 1 0 \cdots 0)$  are rows of  $M$ . Let  $v_0$  be an *extremal* vector of  $\mathbb{P}$ , that is,  $v_0 \in \mathbb{P}$  and there are  $n - 1$  linearly independent rows  $r_1, \dots, r_{n-1}$  of  $M$  such that  $r_i \cdot v_0 = 0$ . We claim that we can rescale  $v_0$  to obtain  $v_1$ , a vector in which each entry is a  $(nk + n \log(n)/2)$ -bounded integer.

To see this, define  $r_0 := (1 \cdots 1)$  and let  $A$  be the matrix with rows  $r_0, r_1, \dots, r_{n-1}$ . Then  $v_0$  is the unique solution to

$$A \cdot v = \|v_0\| \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

By Cramer's rule, if  $A_i$  is the matrix obtained by replacing the  $i^{\text{th}}$  column of  $A$  by  $(1 0 \cdots 0)^T$  then the  $i^{\text{th}}$  entry of  $v_0$  is given by

$$\|v_0\| \cdot \frac{\det(A_i)}{\det(A)}.$$

Hence, by rescaling  $v_0$  by  $|\det(A)|/\|v_0\|$  we obtain a vector  $v_1 \in \mathbb{P}$  whose  $i^{\text{th}}$  entry is  $|\det(A_i)|$ . Moreover  $v_1$  is  $(nk + n \log(n)/2)$ -bounded as each  $|\det(A_i)|$  is by Lemma 3.2.1.  $\square$

### 3.3 Bounds on reducing curves

We start by rephrasing Lemma 2.1.2 as a linear programming problem.

**Lemma 3.3.1.** *There are  $O(1)$ -bounded  $\zeta \times 3\zeta$  matrices  $F_1, \dots, F_k$  such that a vector  $v \in \mathbb{N}^\zeta$  corresponds to a multicurve  $\gamma \in \mathcal{C}(S)$  if and only if  $v \neq 0$  and*

$$F_i \cdot v \geq 0$$

for some  $i$ .  $\square$

Similarly, we express Lemma 2.2.2 using matrices. A mapping class induces a piecewise-linear transformation on the weight space of a triangulation. We record this transformation via two collections of matrices  $\{A_i\}$  and  $\{B_i\}$ . Here the matrix  $A_i$  describes the linear transformation inside of the  $i^{\text{th}}$  cell and the matrix  $B_i$  describes the system of linear inequalities which define the  $i^{\text{th}}$  cell.

For a path consisting of a single reordering of the edges of a triangulation we define its matrices to be:

$$\begin{aligned} A_1 &:= \text{the permutation matrix of the reordering} \\ B_1 &:= (0 \cdots 0) \end{aligned}$$

For a path consisting of a single flip of an edge  $e$  of a triangulation, as shown in Figure 2.1, we define its matrices to be:

$$\begin{aligned} A_1 &:= \text{Id} + E_{ea} + E_{ec} - 2E_{ee} \\ A_2 &:= \text{Id} + E_{eb} + E_{ed} - 2E_{ee} \\ B_1 &:= E_a + E_c - E_b - E_d \\ B_2 &:= E_b + E_d - E_c - E_a \end{aligned}$$

where  $E_i$  is the  $\zeta \times 1$  matrix with a 1 at position  $(i, 1)$  and 0 everywhere else and  $E_{ij}$  is the  $\zeta \times \zeta$  matrix with a 1 at position  $(i, j)$  and 0 everywhere else.

For a longer path  $p \cdot p'$  we inductively define its matrices as:

$$A_k'' := A_j' \cdot A_i \quad \text{and} \quad B_k'' := \left( \frac{B_i}{B_j' \cdot A_i} \right)$$

where  $\{A_i\}$  and  $\{B_i\}$  are the matrices of  $p$  and  $\{A_j'\}$  and  $\{B_j'\}$  are the matrices of  $p'$ . For further details of this construction see [8, Section 3].

**Lemma 3.3.2.** *Suppose that  $h \in \text{Mod}^+(S)$  is a mapping class and that  $p$  is a path from  $\mathcal{T}$  to  $h(\mathcal{T})$ . Let  $\{A_i\}$  and  $\{B_i\}$  be the matrices defined above.*

1. *Each  $A_i$  and  $B_i$  is  $\ell(p)$ -bounded.*
2. *Each  $B_i$  has  $O(\ell(p))$  rows.*
3. *For each multicurve  $\gamma \in \mathcal{C}(S)$  we have that  $B_i \cdot \mathcal{T}(\gamma) \geq 0$  for some  $i$ .*
4. *For each multicurve  $\gamma \in \mathcal{C}(S)$  we have that*

$$\mathcal{T}(h(\gamma)) = A_i \cdot \mathcal{T}(\gamma) \quad \text{if and only if} \quad B_i \cdot \mathcal{T}(\gamma) \geq 0.$$

□

**Theorem 3.3.3** ([8, Theorem 4.1]). *Suppose that  $h \in \text{Mod}^+(S)$  is a mapping class and that  $p$  is a path from  $\mathcal{T}$  to  $h(\mathcal{T})$ . If  $h$  is reducible then there is an  $h$ -invariant multicurve such that  $\mathcal{T}(\gamma)$  is  $O(\ell(p))$ -bounded.*

*Proof.* Let  $\{A_i\}$  and  $\{B_i\}$  be the matrices of Lemma 3.3.2. Additionally, let  $\{F_j\}$  be the matrices of Lemma 3.3.1. Then for each  $i$  and  $j$ , let

$$M(i, j) := \begin{pmatrix} \frac{A_i - \text{Id}}{-(A_i - \text{Id})} \\ \frac{B_i}{F_j} \\ \text{Id} \end{pmatrix}$$

We begin by claiming that  $h$  is reducible if and only if there is a non-trivial solution to  $M(i, j) \cdot v \geq 0$  for some  $i$  and  $j$ .

To prove this claim, firstly suppose that  $h(\gamma) = \gamma$  and let  $v := \mathcal{T}(\gamma) \neq 0$ . Let  $i$  be such that  $B_i \cdot v \geq 0$  and so  $A_i \cdot v = \mathcal{T}(h(\gamma)) = v$ . Hence

$$(A_i - \text{Id}) \cdot v \geq 0 \quad \text{and} \quad -(A_i - \text{Id}) \cdot v \geq 0.$$

As  $\gamma$  is a multicurve there is a  $j$  such that  $F_j \cdot v \geq 0$ . Thus  $v$  is a non-trivial vector and  $M(i, j) \cdot v \geq 0$ .

Conversely, suppose that  $v$  is a non-trivial solution to  $M(i, j) \cdot v \geq 0$ . Without loss of generality we may assume that the entries of  $v$  are non-negative and rational as  $M(i, j)$  defines a rational polytope. Furthermore, by scaling  $v$  by a sufficiently large natural number we may assume that

$$F_j \cdot v \geq_2 0.$$

Hence, there is a multicurve  $\gamma \in \mathcal{C}(S)$  such that  $\mathcal{T}(\gamma) = v$ . As  $B_i \cdot v \geq 0$  and  $v$  lies in the kernel of  $A_i - \text{Id}$ , we have that  $h(\gamma) = \gamma$ . This proves the claim.

Now each  $A_i$  and  $B_i$  is  $O(\ell(p))$ -bounded and so each  $M(i, j)$  is too. Therefore by Proposition 3.2.2, there is a  $O(\ell(p))$ -bounded vector  $v_0$  such that  $M(i, j) \cdot v_0 \geq 0$ . Then

$$F_j \cdot 2v_0 \geq_2 0.$$

Thus there is a multicurve  $\gamma \in \mathcal{C}(S)$  such that  $\mathcal{T}(\gamma) = 2v_0$ . Hence,  $\gamma$  is an  $h$ -invariant multicurve such that  $\mathcal{T}(\gamma)$  is  $O(\ell(p))$ -bounded.  $\square$

By applying this theorem to the standard path of a word we immediately obtain that:

**Corollary 3.3.4.** *Fix  $S$ , a marked surface, and  $X$ , a finite generating set of  $\text{Mod}^+(S)$ . If  $h \in X^*$  is reducible then there is an  $h$ -invariant multicurve which is  $O(\ell(p))$ -bounded.*  $\square$

We may use such a multicurve as a certificate that  $h \in X^*$  is reducible. Given its vector, by Lemma 2.1.2 we can first check that it corresponds to a multicurve  $\gamma$  in  $O(\ell(h))$  operations. Secondly, by Lemma 2.2.2 we can compute  $\mathcal{T}_*(h(\gamma))$  in  $O(\ell(h)^2)$  time. Finally we can verify that  $\mathcal{T}_*(h(\gamma)) = \mathcal{T}_*(\gamma)$  and so check that  $h$  is reducible in  $O(\ell(h))$  time.

**Corollary 3.3.5.** *Fix  $S$ , a marked surface, and  $X$ , a finite generating set of  $\text{Mod}^+(S)$ . Deciding whether a word in  $X^*$  is reducible is a problem in **NP**.*  $\square$

### 3.4 Subsurfaces

When  $h \in \text{Mod}^+(S)$  is a reducible mapping class, as well as fixing a multicurve it also fixes a proper subsurface. In order to study the induced mapping class on such an invariant subsurface without talking about surfaces with boundary, we introduce the notion of *crushing*  $S$  along a multicurve  $\gamma \in \mathcal{C}(S)$ .

**Definition 3.4.1.** We *crush*  $S$  along  $\gamma$  to obtain the (possibly disconnected) surface  $S_\gamma$  by:

1. removing an open regular neighbourhood of  $\gamma$ ,
2. collapsing the new boundary components to additional marked points, and then
3. removing any components that are twice marked spheres.

See Figure 3.1 for example.

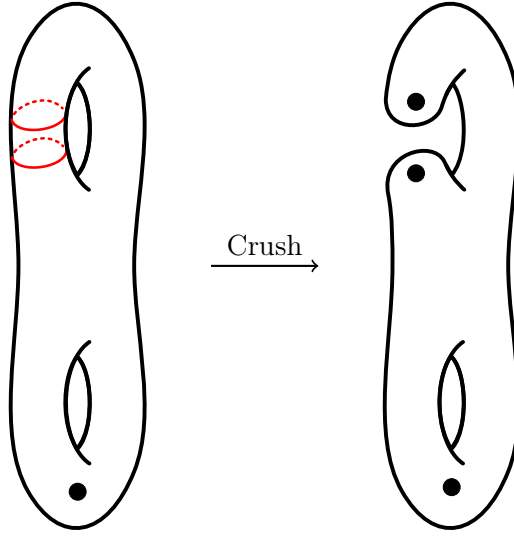


Figure 3.1: Crushing along a multicurve.

Now if  $\mathcal{T}$  is a triangulation of  $S$  then we may track it as we crush  $S$  along a multicurve  $\gamma \in \mathcal{C}(S)$ . After collapsing any bigons that are created, this results in a triangulation  $\mathcal{T}_\gamma$  of  $S_\gamma$ . There is a canonical bijection between the edges of  $\mathcal{T}$  and  $\mathcal{T}_\gamma$  and so  $\zeta(S_\gamma) = \zeta$ . To see this consider the following construction of  $\mathcal{T}_\gamma^*$ , the dual graph of  $\mathcal{T}_\gamma$ :

1. For each face  $f \in F(\mathcal{T})$ , place a vertex  $v$  in the *core* of  $f$ , that is, the component of  $f - \gamma$  which meets all three sides of  $f$ .
2. Extend three half-edges from  $v$  to  $\partial f$  whilst avoiding  $\gamma$ .
3. Extend these half edges along the *corridors* created by parallel strands of  $\gamma$  until they connect with another half edge.

See Figure 3.2.

Most importantly for us, Erickson and Nayyeri showed how to find the ends of each of the corridors of  $\gamma$  in polynomial time.

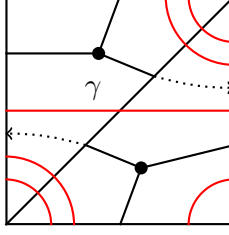


Figure 3.2: Building the dual graph of  $\mathcal{T}_\gamma$  inside  $S - \gamma$ .

**Theorem 3.4.2** ([35]). *Suppose that  $\mathcal{T}(\gamma)$  is  $k$ -bounded. There is an algorithm to compute  $\mathcal{T}_\gamma$  in  $O(\text{poly}(k))$  operations.*  $\square$

**Proposition 3.4.3.** *If  $p$  is a path from  $\mathcal{T}$  to  $\mathcal{T}'$  then crushing each triangulation of  $p$  along  $\gamma$ , and possibly discarding any repeated triangulations, gives a path  $p_\gamma$  from  $\mathcal{T}_\gamma$  to  $\mathcal{T}'_\gamma$ .*

*Proof.* The result clearly holds when  $p$  consists of a single reordering of the edges of  $\mathcal{T}$ . If  $p$  consists of a single flip then the combinatorics of  $\mathcal{T}_\gamma$  and  $\mathcal{T}'_\gamma$  agree away from the faces coming from the faces incident to the flipped edge. Thus  $\mathcal{T}_\gamma$  and  $\mathcal{T}'_\gamma$  share at least  $\zeta - 1$  edges and so they are either equal or differ by a single flip. The result then follows for all paths by induction on  $\ell(p)$ .  $\square$

In fact when  $\mathcal{T}'$  is obtained by flipping the edge  $e$  of  $\mathcal{T}$ , we have that  $\mathcal{T}_\gamma$  and  $\mathcal{T}'_\gamma$  are equal if and only if there is an arc of  $\gamma$  passing from one side of the square containing  $e$  to the opposite side. This occurs if and only if, following the notation of Figure 2.1,  $\iota(\gamma, a) + \iota(\gamma, c) \neq \iota(\gamma, b) + \iota(\gamma, d)$ .

We note that by construction  $\ell(p_\gamma) \leq \ell(p)$ .

**Corollary 3.4.4.** *Suppose that  $p$  is a path from  $\mathcal{T}$  to  $\mathcal{T}'$  and  $\gamma \in \mathcal{C}(S)$  is a  $k$ -bounded multicurve. If  $\mathcal{T}(\gamma)$  is  $k$ -bounded then we can compute  $p_\gamma$  in  $O(\ell(p) \text{poly}(k))$  operations.*  $\square$

### 3.4.1 Bounds on maximal curves

We write  $h_\gamma \in \text{Mod}^+(S_\gamma)$  for the mapping class induced on  $S_\gamma$  by  $h$ . Using this notation, if  $p$  is a path from  $\mathcal{T}$  to  $h(\mathcal{T})$  then  $p_\gamma$  is a path from  $\mathcal{T}_\gamma$  to  $h_\gamma(\mathcal{T}_\gamma)$ .

**Definition 3.4.5.** A multicurve  $\gamma \in \mathcal{C}(S)$  is  $h$ -maximal if it is  $h$ -invariant and  $h_\gamma$  is irreducible.

Now the bijection between edges of  $\mathcal{T}$  and the edges of  $\mathcal{T}_\gamma$  gives a map  $\iota_\gamma: \mathcal{C}(S_\gamma) \rightarrow \mathcal{C}(S)$ , lifting multicurves on  $S_\gamma$  back into  $S$ . Furthermore, if  $\mathcal{T}(\gamma)$  is

$k$ -bounded then there is a  $k$ -bounded integer matrix  $M$  such that

$$\mathcal{T}(\iota_\gamma(\gamma')) = M \cdot \mathcal{T}_\gamma(\gamma').$$

However, it will be easier to work with the map:

$$\overline{\iota}_\gamma: \mathcal{C}(S_\gamma) \rightarrow \mathcal{C}(S) \quad \text{given by} \quad \overline{\iota}_\gamma(\gamma') := \iota_\gamma(\gamma') \cup \gamma.$$

**Lemma 3.4.6.** *Suppose that  $\mathcal{T}(\gamma)$  is  $k$ -bounded. If  $\gamma' \in \mathcal{C}(S_\gamma)$  is multicurve and  $\mathcal{T}_\gamma(\gamma')$  is  $k'$ -bounded then  $\mathcal{T}(\overline{\iota}_\gamma(\gamma'))$  is  $(k + k' + \zeta)$ -bounded.  $\square$*

We may repeat the construction of an invariant multicurve on  $S_\gamma$  and use this bound to control the complexity of the result we obtain back on  $S$ . To help us do this rigorously we introduce a second notion of complexity, closely related to the dimension of the curve complex of  $S_\gamma$  [59].

**Definition 3.4.7.** The *complexity* of a multicurve  $\gamma \in \mathcal{C}(S)$  is

$$\xi(\gamma) := 3g(S_\gamma) + n(S_\gamma) - 3|S_\gamma|.$$

Now note that if  $\gamma \in \mathcal{C}(S)$  and  $\gamma' \in \mathcal{C}(S_\gamma)$  then

$$\xi(\overline{\iota}_\gamma(\gamma')) < \xi(\gamma).$$

Additionally,  $\xi(\gamma) \leq \zeta$  and if  $\xi(\gamma) = 0$  then  $\mathcal{C}(S_\gamma) = \emptyset$  and so  $\gamma$  must be  $h$ -maximal.

**Theorem 3.4.8.** *Suppose that  $h \in \text{Mod}^+(S)$  is a reducible mapping class and that  $p$  is a path from  $\mathcal{T}$  to  $h(\mathcal{T})$ . Then there is an  $h$ -maximal multicurve  $\gamma \in \mathcal{C}(S)$  such that  $\mathcal{T}(\gamma)$  is  $O(\ell(p))$ -bounded.*

*Proof.* As  $h$  is reducible there is an  $h$ -invariant multicurve  $\gamma \in \mathcal{C}(S)$  such that  $\mathcal{T}(\gamma)$  is  $O(\ell(p))$ -bounded by Theorem 3.3.3.

Now suppose that  $\gamma$  is not  $h$ -maximal. As  $h_\gamma$  is reducible, we can reapply Theorem 3.3.3 to the crushed path  $p_\gamma$  from  $\mathcal{T}_\gamma$  to  $h_\gamma(\mathcal{T}_\gamma)$ . As  $\ell(p_\gamma) \leq \ell(p)$ , we deduce that there is an  $h_\gamma$ -invariant multicurve  $\gamma' \in \mathcal{C}(S_\gamma)$  such that  $\mathcal{T}_\gamma(\gamma')$  is  $O(\ell(p))$ -bounded.

Following this we redefine  $\gamma$  to be  $\overline{\iota}_\gamma(\gamma')$ . This is again an  $h$ -invariant multicurve and, by Lemma 3.4.6, is still  $O(\ell(p))$ -bounded. However, doing this decreases  $\xi(\gamma)$  and so after repeating this process at most  $\zeta$  times  $\gamma$  must become  $h$ -maximal.  $\square$



Again, by applying this theorem to the standard path of a word we obtain that:

**Corollary 3.4.9.** *If  $h \in X^*$  is reducible then there is an  $h$ -maximal multicurve which is  $O(\ell(h))$ -bounded.*  $\square$

### 3.4.2 The canonical curve system

The *canonical curve system*  $\sigma(h) \in \mathcal{C}(S)$  of a mapping class  $h \in \text{Mod}^+(S)$  is the intersection of all  $h$ -maximal multicurves [38, Page 373]. It is non-empty if and only if the mapping class is reducible and of infinite order [52, Theorem 4.44].

Koberda and Mangahas showed there is an exponential upper bound on the entries of  $\mathcal{T}(\sigma(h))$  [54, Theorem 1.1]. Corollary 3.4.9 also provides an alternate proof of their theorem.

**Proposition 3.4.10** ([8, Corollary 5.6]). *Suppose that  $h \in \text{Mod}^+(S)$  is a mapping class and that  $p$  is a path from  $\mathcal{T}$  to  $h(\mathcal{T})$ . Then  $\mathcal{T}(\sigma(h))$  is  $O(\ell(p))$ -bounded.*

*Proof.* If  $\sigma(h)$  is empty then the result holds trivially. Otherwise,  $h$  is reducible and so by Corollary 3.4.9 there is an  $h$ -maximal multicurve  $\gamma$  which is  $O(\ell(h))$ -bounded. Therefore, as  $\sigma(h) \subseteq \gamma$  it must be  $O(\ell(h))$ -bounded too.  $\square$

Applying this to the standard path of a word gives:

**Corollary 3.4.11.** *Fix  $S$ , a marked surface, and  $X$ , a finite generating set of  $\text{Mod}^+(S)$ . For each word  $h \in X^*$ , the canonical curve system  $\sigma(h)$  is  $O(\ell(h))$ -bounded.*  $\square$

## Chapter 4

# Pseudo-Anosov mapping classes

In this chapter we generalise the notion of the weight space of a triangulation from  $\mathbb{N}^\zeta$  to  $\mathbb{R}_{\geq 0}^\zeta$ . This allows us to combinatorially represent and manipulate measured laminations on the surface; after first extending our model of computation to allow us to manipulate numbers with finitely many decimal places. We recall the notion of a measured train track and the maximal splitting move, originally described by Agol. We construct an explicit upper bound on the number of these moves needed to reach a particularly nice train track, allowing us to show that deciding whether a mapping class is pseudo-Anosov is a problem in **NP**.

Much of the material of this chapter is based on an earlier preprint [9]. However many of the bounds have been significantly improved. For example, the periodic length bound of Section 4.2.9 is now independent of the quasi-geodesic constant of [60, Theorem 6.2] and the quantity  $h_0$  of the main algorithm has been strengthened to also be a polynomial function of  $\zeta$ .

### 4.1 Measured laminations

For a more detailed discussion of measured laminations and the topology of  $\mathcal{ML}(S)$  we refer the reader to [87, Chapter 8] and [72, Chapter 2].

**Definition 4.1.1** ([75, Section 1.7] [27, Section 3] [63] [87, Definition 8.5.2]). Choose a hyperbolic metric on  $S - P$ . A *measured lamination*  $\mathcal{L}$  is:

- a closed subset  $A \subseteq S$  foliated by simple geodesics called *leaves*, together with
- a transverse ( $\sigma$ -additive) measure that:
  - has *full support*, that is, any arc meeting the leaves of  $A$  transversely is assigned a non-zero measure, and

– is translation invariant.

Up to isotopy, this construction is independent of the choice of metric [72, Page 150]. Thus we may think of measured laminations as objects that are only defined up to isotopy. Therefore, for a measured lamination  $\mathcal{L}$  we define the measure it assigns to an isotopy class  $\alpha$  to be

$$\mathcal{L}(\alpha) := \inf_{a \in \alpha} \mathcal{L}(a).$$

We denote the space of measured laminations on  $S$  by  $\mathcal{ML}(S)$ . Again, to avoid a recurring edge case we will disallow the *empty measured lamination*, the measured lamination that assigns zero measure to every curve, throughout.

Multicurves may be thought of as measured laminations where the foliated subset is the multicurve and the measure is given by the geometric intersection number. Many of the results about multicurves from the previous chapter generalise to laminations. For example, just as a multicurve is determined by its intersection numbers with the edges  $e_1, \dots, e_\zeta$  of a triangulation, a measured lamination is determined by the measure it assigns to  $e_1, \dots, e_\zeta$ . Again, the triangulation  $\mathcal{T}$  provides a coordinate system on  $\mathcal{ML}(S)$  where the measured lamination  $\mathcal{L} \in \mathcal{ML}(S)$  corresponds to the point:

$$\mathcal{T}(\mathcal{L}) := \begin{pmatrix} \mathcal{L}(e_1) \\ \vdots \\ \mathcal{L}(e_\zeta) \end{pmatrix} \in \mathbb{R}_{\geq 0}^\zeta$$

Once more, although distinct laminations have distinct coordinates, not all coordinates correspond to measured laminations. In fact  $(v_1 \cdots v_\zeta)^T$  corresponds to a measured lamination if and only if:

- $\sum v_i > 0$ ,
- for each triangle  $a, b, c$  it satisfies the triangle inequality  $v_a + v_b \geq v_c$ , and
- for each vertex  $v$  there is a face with sides  $a, b, c$  such that  $v \subseteq a \cap b$  and  $v_a + v_b = v_c$ .

Many of the results of Chapter 2 have variants that hold for laminations. For example, the following proposition generalises Proposition 2.2.1 to laminations.

**Proposition 4.1.2** ([67, Page 30]). *Suppose that  $\mathcal{L}$  is a measured lamination and  $e$  is a flippable edge of a triangulation  $\mathcal{T}$  as shown in Figure 2.1 then*

$$\mathcal{L}(e') = \max(\mathcal{L}(a) + \mathcal{L}(c), \mathcal{L}(b) + \mathcal{L}(d)) - \mathcal{L}(e). \quad \square$$

Again the mapping class group  $\text{Mod}^+(S)$  has a natural action on  $\mathcal{ML}(S)$ . Following Proposition 4.1.2, this action is again piecewise linear with respect to the coordinate system induced on  $\mathcal{ML}(S)$  by a triangulation. From this we obtain a generalisation of Lemma 3.3.2.

**Lemma 4.1.3.** *Suppose that  $h \in \text{Mod}^+(S)$  is a mapping class and that  $p$  is a path from  $\mathcal{T}$  to  $h(\mathcal{T})$ . Let  $\{A_i\}$  and  $\{B_i\}$  be the matrices of Lemma 3.3.2.*

1. *for each lamination  $\mathcal{L} \in \mathcal{ML}(S)$  we have that  $B_i \cdot \mathcal{T}(\mathcal{L}) \geq 0$  for some  $i$ .*
2. *For each lamination  $\mathcal{L} \in \mathcal{ML}(S)$  we have that*

$$\mathcal{T}(h(\mathcal{L})) = A_i \cdot \mathcal{T}(\mathcal{L}) \quad \text{if and only if} \quad B_i \cdot \mathcal{T}(\mathcal{L}) \geq 0. \quad \square$$

**Definition 4.1.4** ([39, Exposé 12] [27, Page 95]). A mapping class  $h \in \text{Mod}^+(S)$  is *pseudo-Anosov* if there is a measured lamination  $\mathcal{L} \in \mathcal{ML}(S)$  which is:

- *projectively invariant*, that is,  $h(\mathcal{L}) = \lambda \cdot \mathcal{L}$  for some  $\lambda \in \mathbb{R}_{\geq 0}$ , and
- *filling*, that is, it assigns positive measure to every multicurve.

A word  $h \in X^*$  is *pseudo-Anosov* if its corresponding mapping class is.

Furthermore, if  $h$  is pseudo-Anosov then, up to rescaling the measure by a positive constant, there is a unique projectively invariant, filling measured lamination which is *stable*, that is,  $\lambda > 1$ . To minimise the number of adjectives we will refer to this simply as the *stable* laminations of  $h$ , denoted by  $\mathcal{L}^+(h)$ .

**Problem 4.1.5** (The pseudo-Anosov problem). Fix  $S$ , a marked surface, and  $X$ , a finite generating set of  $\text{Mod}^+(S)$ . Given a word  $h \in X^*$  decide whether  $h$  is pseudo-Anosov.

The techniques given in Chapter 1 show that when the marked surface is  $S_{1,1}$  or  $S_{0,4}$  the pseudo-Anosov problem lies in **P**. Furthermore Menzel and Parker showed that for  $S_{1,2}$  the pseudo-Anosov problem lies in **NP** [65]. Before discussing this problem for an arbitrary marked surface, we remark that filling laminations act similarly to the non-isolating curves of Section 2.4.2. For example, directly following the proof of Lemma 2.4.3 gives:

**Proposition 4.1.6.** *Suppose that  $\mathcal{T}$  is a triangulation. If  $\mathcal{L}$  is a filling lamination then there is a flippable edge  $e$  of  $\mathcal{T}$  such that*

$$\mathcal{L}(\mathcal{T}') < \mathcal{L}(\mathcal{T})$$

where  $\mathcal{T}'$  is the triangulation obtained by flipping  $e$ . □

## 4.2 Train tracks

The hardest part of the pseudo-Anosov problem is validating that a projectively invariant lamination is filling. To tackle this we introduce the notion of a train track. For a detailed discussion of train tracks see [75] and [77].

**Definition 4.2.1** ([75, Section 1.1]). A *measured train track (representing  $\mathcal{L}$ )* is a pair  $T = (\tau, \mu)$  consisting of:

- a *train track*  $\tau$ , that is, a trivalent graph on  $S$  (whose vertices we refer to as *switches* and edges we refer to as *branches*) with a well defined tangency at each switch such that there is no switch with all branches emanating from it in the same direction and no complementary region of  $\tau$  is a nullogon, monogon, bigon, once-marked nullogon or annulus, and
- a *transverse measure*  $\mu$  such that there is a smooth map  $\phi: S \rightarrow S$ , isotopic to the identity, such that:

- $\phi(\mathcal{L}) = \tau$ ,
- $\phi|_{\mathcal{L}}: \mathcal{L} \rightarrow \tau$  is a submersion, and
- $\mu = \mathcal{L} \circ \phi^{-1}$

**Definition 4.2.2** ([75, Page 119]). If  $T = (\tau, \mu)$  is a measured train track then we may *split* along one of its branches  $e$  to obtain a new measured train track  $T' = (\tau', \mu')$  as shown in Figure 4.1.

The *graph of measured train tracks (representing  $\mathcal{L}$ )* is a graph whose vertices are measured train tracks (representing  $\mathcal{L}$ ) and there is an (unoriented) edge from  $T$  to  $T'$  if and only if  $T'$  can be obtained by splitting some non-empty subset of the branches of  $T$  each of which has the same transverse measure. This graph is connected [75, Theorem 2.8.5] and we write  $d(T, T')$  for the distance between two measured train tracks when each edge of this graph is assigned length one.

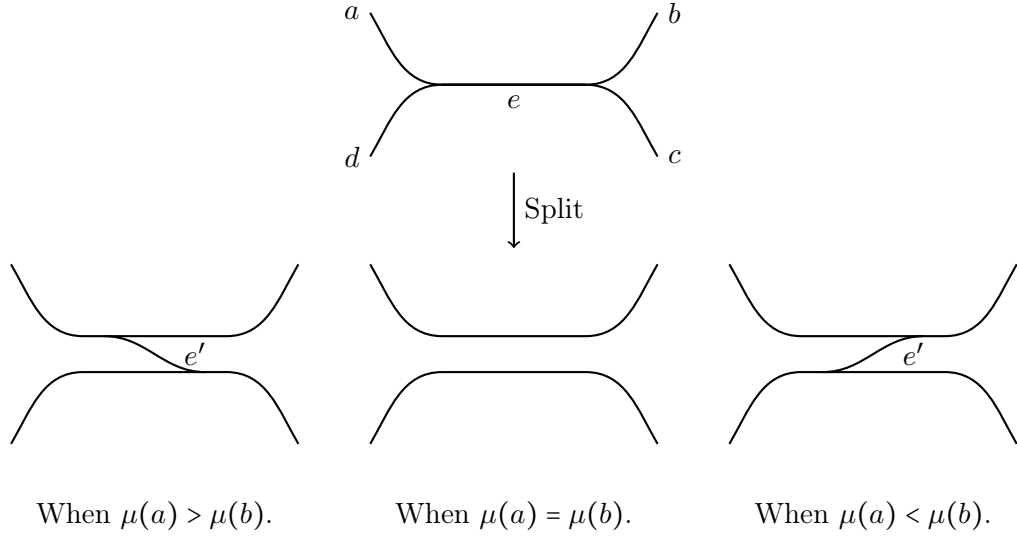


Figure 4.1: The possibilities for splitting a branch  $e$ . In both cases  $\mu(e') = |\mu(a) - \mu(b)| = |\mu(c) - \mu(d)|$  [75, Figure 2.1.2].

**Definition 4.2.3** ([3, Section 2]). The *maximal splitting* of a measured train track  $T$  is the measured train track  $s(T)$  obtained by simultaneously splitting all branches of maximal transverse measure.

**Lemma 4.2.4.** *If  $T$  and  $T'$  are measured train tracks (representing  $\mathcal{L}$ ) then*

$$d(s(T), s(T')) \leq d(T, T'),$$

*that is,  $s$  is a non-expansive map.*

*Proof.* Begin by considering the case in which  $d(T, T') = 1$ . Without loss of generality, there is a subset  $B$  of the branches of  $T$  such that by splitting the branches in  $B$  we obtain  $T'$  and each branch in  $B$  has the same transverse measure. Let  $M$  be the subset of the branches of  $T$  with maximal transverse measure. There are three possible cases to consider:

1. If  $B = M$  then  $s(T) = T'$  and so  $d(s(T), s(T')) = d(T', s(T')) \leq 1$ .
2. If  $B \subset M$  then the branches in  $M - B$  also appear in  $T'$ . It is these branches that are split when maximally splitting  $T'$  and so  $s(T) = s(T')$ . Hence  $d(s(T), s(T')) = 0$ .
3. If  $B \not\subset M$  then the branches in  $B$  also appear in  $s(T)$ . Splitting these branches results in  $s(T')$  as  $M$  and  $B$  lie in disjoint open sets on  $S$ . Hence  $d(s(T), s(T')) = 1$ .

In any case  $d(s(T), s(T')) \leq 1$  and so the result holds by induction on  $d(T, T')$ .  $\square$

#### 4.2.1 Maximal splittings and the axis

Now suppose that there is a mapping class  $h \in \text{Mod}^+(S)$  such that

$$h(\mathcal{L}) = \lambda \cdot \mathcal{L}$$

for some  $\lambda > 1$ . In this case, Agol showed that the sequence of train tracks obtained by repeatedly performing maximal splittings is eventually periodic.

**Theorem 4.2.5** ([3, Theorem 3.5]). *If  $T$  is a measured train track representing  $\mathcal{L}$  such that*

$$h(\mathcal{L}) = \lambda \cdot \mathcal{L}$$

*then there exists  $m, n \in \mathbb{N}$  such that  $s^{m+n}(T) = \widehat{h}(s^n(T))$  where*

$$\widehat{h}(\tau, \mu) := (h(\tau), \mu/\lambda). \quad \square$$

We refer to the smallest such  $m$  and  $n$  as the *periodic* and *preperiodic lengths* of  $T$  respectively. We note that  $m$  depends only on  $\mathcal{L}$  and is independent of  $T$ .

**Definition 4.2.6.** The *axis* of  $\mathcal{L}$  is the bi-infinite sequence of measured train tracks  $A = A(\mathcal{L}) := \{\mathbf{T}_i\}_{i=-\infty}^{\infty}$  such that

$$s(\mathbf{T}_i) = \mathbf{T}_{i+1} \quad \text{and} \quad \widehat{h}(\mathbf{T}_i) = \mathbf{T}_{i+m}.$$

The measured train tracks on the axis are useful as you can determine whether  $\mathcal{L}$  is filling purely from the combinatorics of their underlying train tracks.

**Definition 4.2.7.** A measured train track  $T = (\tau, \mu)$  is *filling* if every complementary region of  $\tau$  is either a disk or a once-marked disk.

**Lemma 4.2.8.** *The measured lamination  $\mathcal{L}$  is filling if and only if  $\mathbf{T}_i$  is.*

*Proof.* If  $\mathbf{T}_i = (\tau, \mu)$  is not filling then there is a curve  $\gamma \in \mathcal{C}(S)$  such that  $\gamma \cap \tau = \emptyset$ . Therefore  $\mathcal{L}(\gamma) = 0$  and so  $\mathcal{L}$  is not filling.

Conversely, if  $\mathcal{L}$  is not filling then there is a curve  $\gamma$  such that  $\mathcal{L}(\gamma) = 0$ . There is a measured train track  $T = (\tau, \mu)$ , representing  $\mathcal{L}$ , such that  $\tau$  and  $\gamma$  are disjoint. By Theorem 4.2.5 there are  $j$  and  $k$  such that  $\widehat{h}^k(s^j(T)) = \mathbf{T}_i$ . As  $T$  is not filling and this is preserved by both maximal splittings and homeomorphisms,  $\mathbf{T}_i$  is not filling either.  $\square$

### 4.2.2 Getting to the axis

We decide whether  $\mathcal{L}$  is filling by constructing a measured train track on the axis  $A$ . To do this we give upper bounds for the periodic and preperiodic lengths of  $T$ .

**Theorem 4.2.9.** *The periodic length of  $T$  is at most  $d(T, \widehat{h}(T))$ .*

*Proof.* Suppose that

$$p := T_0, T_1, \dots, T_k$$

is an (unparameterised) path from  $T$  to  $\widehat{h}(T)$  of length  $d(T, \widehat{h}(T))$ , that is, a sequence of measured train tracks such that  $d(T_i, T_{i+1}) \leq 1$ . Let  $T_i^j := s^j(T_i)$  and note that for each  $j$ :

- $\widehat{h}(T_0^j) = T_k^j$ , and
- $p_j := T_0^j, \dots, T_k^j$  is also an (unparameterised) path as  $s$  is non-expansive (Lemma 4.2.4).

When  $j$  is sufficiently large, by Theorem 4.2.5 each  $T_i^j$  lies on the axis  $A$  and therefore so does  $p_j$ . The endpoints of  $p_j$  must be  $\mathbf{T}_{k'}$  and  $\mathbf{T}_{k'+m}$  and so we conclude that  $m \leq k = d(T, \widehat{h}(T))$ .  $\square$

To bound the preperiodic length, the following theorem of Tao is key. In particular, note that the constant depends only on the underlying surface and is independent of the chosen mapping class  $h$  and measured lamination  $\mathcal{L}$ .

**Theorem 4.2.10** ([84, Theorem B], [59, Theorem 7.2]). *There is a constant  $K = K(S)$  such that if  $T$  and  $T'$  are measured train tracks (representing  $\mathcal{L}$ ) then there is a measured train track  $T''$  in the orbit of  $T'$  under  $h$  such that*

$$d(T, T'') \leq K(d(T, \widehat{h}(T)) + d(T', \widehat{h}(T'))). \quad \square$$

Agol showed that for any measured train track  $T$  (representing  $\mathcal{L}$ ), if we repeatedly perform maximal splittings then eventually every branch of  $T$  split [3, Lemma 2.1]. For train tracks on the axis of  $\mathcal{L}$  we can give an explicit upper bound on the number of maximal splittings needed.

**Lemma 4.2.11.** *If  $\mathbf{T}_i$  is a train track on the axis of  $\mathcal{L}$  then every branch of  $\mathbf{T}_i$  must be split within  $3\zeta m$  maximal splittings.*

*Proof.* Let  $B_k$  be the set of branches of  $\mathbf{T}_i$  that are split within  $km$  maximal splittings. Note that  $B_{k+1} = B_k \cup \widehat{h}(B_k)$ . Therefore if  $B_k$  does not contain all branches of



$\mathbf{T}_i$  and  $|B_{k+1}| = |B_k|$  then there would be a branch that is never split, which cannot happen [3, Lemma 2.1]. Hence either  $B_k$  consists of all branches of  $\mathbf{T}_i$  or  $|B_{k+1}| > |B_k|$ . As  $\mathbf{T}_i$  has at most  $3\zeta$  branches, the latter case cannot occur when  $k \geq 3\zeta$ . Thus every branch of  $\mathbf{T}_i$  must be split within  $3\zeta m$  maximal splittings.  $\square$

We note that by this result every branch of  $\mathbf{T}_i$  must become a branch of maximal transverse measure within  $3\zeta m$  maximal splittings.

**Theorem 4.2.12.** *The preperiodic length of  $T$  is at most  $6\zeta K d(T, \widehat{h}(T))^2$ .*

*Proof.* We claim that if  $\mathbf{T}_i$  is a measured train track on  $A$  then the preperiodic length of a measured train track  $T$  is at most  $3\zeta m d(T, \mathbf{T}_i)$ . We prove this by induction on the distance. There is nothing to show in the base case when  $d(T, \mathbf{T}_i) = 0$  so suppose that the claim is true whenever  $d(T, \mathbf{T}_i) < k$ . Now if  $d(T, \mathbf{T}_i) = k$  then let  $T'$  be such that

$$d(T', T) = 1 \quad \text{and} \quad d(T', \mathbf{T}_i) = k - 1$$

as shown in Figure 4.2. Then by induction the preperiodic length of  $T'$  is at most  $3\zeta m(k - 1)$  and so  $\mathbf{T}_j := s^{3\zeta m(k-1)}(T')$  is on the axis  $A$ . For ease of notation let  $T'' := s^{3\zeta m(k-1)}(T)$ , then as  $s$  is non-expansive (Lemma 4.2.4)

$$d(\mathbf{T}_j, T'') \leq 1.$$

There are now three possibilities that can occur.

The first possibility is that  $d(\mathbf{T}_j, T'') = 0$  and so  $T'' = \mathbf{T}_j$ .

The second possibility is that  $d(\mathbf{T}_j, T'') = 1$  and there is a subset  $B$  of the branches of  $\mathbf{T}_j$  such that every branch in  $B$  has the same transverse measure and splitting along these yields  $T''$ . Now consider  $s^p(T'')$  and  $s^p(\mathbf{T}_j)$  where, by Lemma 4.2.11, we choose  $p < 3\zeta m$  such that the branches in  $B$  appear in  $s^p(\mathbf{T}_j)$  with maximal measure. If there are no other branches of maximal measure then maximally splitting  $s^p(\mathbf{T}_j)$  results in  $s^p(T'')$ . Hence  $s^p(T'') = \mathbf{T}_{j+p+1}$ . Otherwise maximally splitting  $s^p(\mathbf{T}_j)$  factors through  $s^p(T'')$  and so  $s^{p+1}(T'') = \mathbf{T}_{j+p+1}$ .

The third possibility is that  $d(\mathbf{T}_j, T'') = 1$  and there is a subset  $B$  of the branches of  $T''$  such that splitting along these yields  $\mathbf{T}_j$ . In this case let  $B'$  be new the branches that are added when the branches in  $B$  are split. Again consider  $s^p(T'')$  and  $s^p(\mathbf{T}_j)$  where, by Lemma 4.2.11, we choose  $p < 3\zeta m$  such that the branches in  $B'$  appear in  $s^p(\mathbf{T}_j)$  with maximal measure. Now in order for  $d(s^{p+1}(\mathbf{T}_j), s^{p+1}(T'')) \leq 1$  we must have already split the branches in  $B$ . Hence  $s^{p+1}(T'')$  must be either  $\mathbf{T}_{j+p+1}$  or  $\mathbf{T}_{j+p}$ .

In any case, it follows that  $s^{3\zeta m k}(T) = s^{3\zeta m}(T'')$  is either:

$$\mathbf{T}_{j+3\zeta m-1}, \quad \mathbf{T}_{j+3\zeta m} \quad \text{or} \quad \mathbf{T}_{j+3\zeta m+1}$$

and so is on the axis  $A$ . Hence the claim is true.

Now by Theorem 4.2.10 there is a  $\mathbf{T}_i$  on the axis  $A$  such that

$$d(T, \mathbf{T}_i) \leq K(d(T, \widehat{h}(T)) + d(\mathbf{T}_i, \widehat{h}(\mathbf{T}_i))) = K(d(T, \widehat{h}(T)) + m).$$

Therefore by the previous claim and Theorem 4.2.9 the preperiodic length of  $T$  is at most

$$3\zeta m K(d(T, \widehat{h}(T)) + m) \leq 6\zeta K d(T, \widehat{h}(T))^2. \quad \square$$

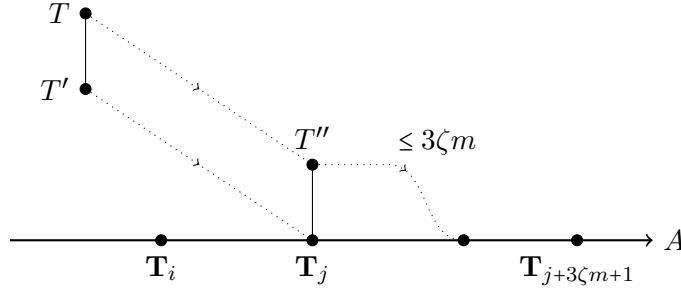


Figure 4.2: Getting to the axis  $A$ .

Combining this with Lemma 4.2.8 we obtain:

**Corollary 4.2.13.** *The measured lamination  $\mathcal{L}$  is filling if and only if  $s^t(T)$  is filling, where  $t := 6\zeta K d(T, \widehat{h}(T))^2$ .  $\square$*

Finally, we note that a triangulation  $\mathcal{T}$  gives rise to a measured train track  $T$  as shown in Figure 4.3. Furthermore if  $p$  is a path from  $\mathcal{T}$  to  $h(\mathcal{T})$  then this descends to a quasi-path in the graph of measured train tracks and so  $d(T, \widehat{h}(T)) \leq 2\ell(p)$ .

### 4.3 The main algorithm

We can now state the main algorithm for validating that a certificate proves that a mapping class is pseudo-Anosov. Again, we use a path  $p$  from  $\mathcal{T}$  to  $h(\mathcal{T})$  to represent  $h$ . We use a *certificate* consisting of decimals  $x_1, \dots, x_\zeta$  and polynomials  $f_1, \dots, f_\zeta \in \mathbb{Z}[x]$ . These decimals represent approximations of the measure assigned to each edge of  $\mathcal{T}$  by the stable lamination of  $h$  while these polynomials are their minimal polynomials.

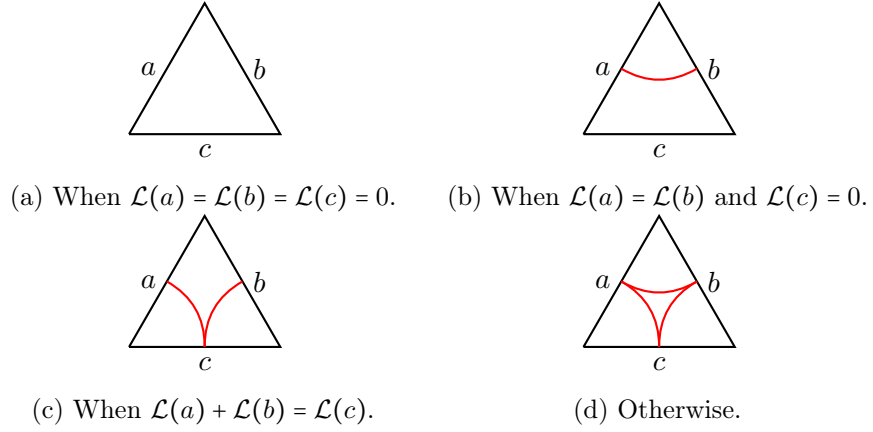


Figure 4.3: A train track coming from a triangulation.

Given a path  $p$  from  $\mathcal{T}$  to  $h(\mathcal{T})$  and a certificate  $x_1, \dots, x_\zeta, f_1, \dots, f_\zeta$ , let:

- $\{A_i\}$  and  $\{B_i\}$  be the matrices of Lemma 3.3.2,
- $t := 24\zeta K \cdot \ell(p)^2$ ,
- $h_0 := \zeta^4(\ell(p) + 6)$ ,
- $h_1 := h_0 + 2\zeta$ ,
- $p_1 := 2\zeta^2(2h_1 + t + 3)$ , and
- $d_1 := p_1 + t + \zeta h_1 + 2$ .

In each of the following stages, all calculations are done to  $d_1$  decimal places and all comparisons are done by *only* comparing the first  $p_1$  decimal places.

1. Check that each  $f_i$  has degree at most  $\zeta$  and the log of the absolute value of each coefficient is at most  $h_0$ .
2. Check that each  $0 \leq x_i \leq 1$ .
3. Check that each pair  $f_i(x_i \pm 10^{-d_1})$  have different signs.
4. For each face of  $\mathcal{T}$  with edges  $a, b, c$  check that  $x_a + x_b \geq x_c$ .
5. For each vertex  $v \in V(\mathcal{T})$  check that there is an incident face of  $\mathcal{T}$  with edges  $a, b, c$  such that  $v \subseteq a \cap b$  and  $x_a + x_b = x_c$ .
6. Check that  $\sum x_i = 1$ .

7. Find  $B_i$  such that  $B_i \cdot (x_1 \cdots x_\zeta)^T \geq 0$  and compute

$$(y_1 \cdots y_\zeta)^T := A_i \cdot (x_1 \cdots x_\zeta)^T \quad \text{and} \quad y := \sum y_i.$$

8. Check that each  $y_i = yx_iy$ .

9. Check that  $y > 1$ .

10. Check that  $s^t(T')$  is filling where  $T'$  is the measured train track corresponding to  $x_1, \dots, x_\zeta$  (see Section 4.2).

We say that a certificate is *accepted* by the main algorithm if every check passes and is *rejected* otherwise.

**Remark 4.3.1.** It may be possible to drop the polynomials  $f_1, \dots, f_\zeta$  from the certificate of  $h$ . It appears that one may modify the main algorithm to first use the Lenstra–Lenstra–Lovász algorithm [56] to recover these polynomials from  $x_1, \dots, x_\zeta$  [29, Section 2.7.2]. However this application of the LLL algorithm relies on certain parameters whose choice “... is subtle, and depends in part on what one knows about the problem” [29, Page 100].

**Remark 4.3.2.** It should be highlighted that the main algorithm relies on the constant  $K(S)$  of Theorem 4.2.10, which is not explicitly known. Hence the main algorithm is not strictly well-defined and cannot be implemented. However only a slight variation is needed to overcome this problem.

Instead of computing the measured train track  $T'$  after  $t$  maximal splittings have been performed, one may compute the maximal splitting sequence  $T_0, T_1, \dots$  until a projectively equal pair are found. Then the lamination is filling if and only if each of these measured train tracks are. Such a projectively equal pair will occur after  $t + m$  maximal splittings which, although unknown, is still a polynomial function of  $\ell(h)$ . Hence the asymptotic running time of this variant is the same as that of the main algorithm.

The added complication of this technique is that the total number of maximal splittings that must be performed is initially unknown. Thus it is possible that after several have been performed the decimal approximations of the edge weights are no longer sufficient to uniquely determine the underlying algebraic numbers. However, if this occurs then one may restart the algorithm with better approximations of these numbers, obtained by using a root-finding algorithm on  $f_i$ .

## 4.4 Algebraic numbers

In order to prove the correctness of the main algorithm, we first recall some results about algebraic numbers.

**Definition 4.4.1** ([88, Section 3.4]). The *height* of a polynomial  $f(x) = \sum a_i x^i \in \mathbb{Z}[x]$  is

$$\text{hgt}(f) := \log(\max(|a_i|)).$$

The *height* of an algebraic number  $\alpha \in \overline{\mathbb{Q}}$  is  $\text{hgt}(\alpha) := \text{hgt}(\mu_\alpha)$  where  $\mu_\alpha \in \mathbb{Z}[x]$  is its minimal integral polynomial.

**Fact 4.4.2.** If  $\alpha, \beta \in \overline{\mathbb{Q}}$  are algebraic numbers then:

- $\text{dg}(\alpha \pm \beta) \leq \text{dg}(\alpha) + \text{dg}(\beta)$ , where  $\text{dg}(\alpha) := \log(\deg(\alpha))$ ,
- $\text{hgt}(\alpha \pm \beta) \leq \text{hgt}(\alpha) + \text{hgt}(\beta) + 1$  [88, Property 3.3],
- $\text{hgt}(\alpha\beta) \leq \text{hgt}(\alpha) + \text{hgt}(\beta)$  [88, Property 3.3],
- $\text{hgt}(\alpha^{-1}) = \text{hgt}(\alpha)$ , and
- if  $\alpha$  is a root of  $f \in \mathbb{Z}[x]$  then  $\text{hgt}(\alpha) \leq \text{hgt}(f) + 2 \deg(f)$  [7, Corollary 10.12].

Perhaps most crucially, an algebraic number of bounded degree and height is uniquely determined by a sufficiently good approximation.

**Lemma 4.4.3** ([7, Lemma 10.2, Lemma 10.3]). *If  $\alpha \neq 0$  then*

$$|\log(|\alpha|)| \leq \text{hgt}(\alpha) + \text{dg}(\alpha).$$

*Alternatively,  $\alpha = 0$  if and only if the integer part of  $\alpha$  is 0 and at least the first  $\text{hgt}(\alpha) + \text{dg}(\alpha)$  decimal places of  $\alpha$  are 0.*  $\square$

**Remark 4.4.4.** In certain circumstances the inequalities of Fact 4.4.2 may be strengthened. For example, if  $\alpha_1, \dots, \alpha_k$  are integers then

$$\text{hgt}\left(\sum_{i=1}^k \alpha_i\right) \leq \max(\text{hgt}(\alpha_i)) + \log(k) \ll \sum \text{hgt}(\alpha_i) + (k-1) \log(2).$$

## 4.5 Correctness

We now prove the correctness of the main algorithm, namely that  $h$  is pseudo-Anosov if and only if there is a certificate that the main algorithm accepts.

### 4.5.1 Acceptance implies pseudo-Anosov

The outline of the first half of the correctness proof is that:

- Stages 1–3 show that each  $x_i$  is close to an algebraic number (Proposition 4.5.1).
- Stages 4–5 show that these algebraic numbers correspond to a measured lamination (Proposition 4.5.2).
- Stage 6 shows that this lamination is *unitary* (with respect to  $\mathcal{T}$ ), that is,  $\|\mathcal{T}(\mathcal{L})\| = 1$ , (Proposition 4.5.3).
- Stage 8 shows that this lamination is projectively invariant under  $h$  (Proposition 4.5.4).
- Stage 9 shows that this lamination is stable (Proposition 4.5.5).
- Stage 10 shows that this lamination is filling (Proposition 4.5.6).

**Proposition 4.5.1.** *If Stages 1–3 of the main algorithm complete then each  $x_i$  lies within  $10^{-d_1}$  of a unique algebraic number  $v_i$  of degree at most  $\zeta$  and height at most  $h_1$ .*

*Proof.* By the intermediate value theorem, Stage 3 shows that  $f_i$  must have a root  $v_i$  in  $[x_i - 10^{-d_1}, x_i + 10^{-d_1}]$ . It follows from Fact 4.4.2 that  $v_i$  is an algebraic number of degree at most  $\deg(f_i) \leq \zeta$  and height at most  $\text{hgt}(f_i) + 2\zeta \leq h_0 + 2\zeta = h_1$ . Finally, any two distinct algebraic number of degree at most  $\zeta$  and height at most  $h_1$  must be separated by at least  $10^{-(2h_1-\zeta)} > 10^{-(d_1-1)}$  and so  $v_i$  is unique.  $\square$

From now on fix  $v_1, \dots, v_\zeta$  to be these algebraic numbers.

**Proposition 4.5.2.** *Stages 4–5 of the main algorithm complete if and only if  $v_1, \dots, v_\zeta$  corresponds to a measured lamination  $\mathcal{L} \in \mathcal{ML}(S)$ , that is,*

$$\mathcal{T}(\mathcal{L}) = (v_1, \dots, v_\zeta).$$

*Proof.* If  $x_a + x_b > x_c$  to  $p_1$  decimal places then

$$v_a + v_b - v_c > 10^{-p_1} - 10^{-d_1} - 10^{-d_1} \geq 10^{-(p_1-1)}.$$

However

$$\text{hgt}(v_a + v_b - v_c) + \deg(v_a + v_b - v_c) \leq 3h_1 + 3\zeta \leq p_1 - 1$$

and so by Lemma 4.4.3 we have that  $v_a + v_b > v_c$ . By the same argument if  $x_a + x_b = x_c$  to  $p_1$  decimal places then  $v_a + v_b = v_c$ . Hence  $v_1, \dots, v_\zeta$  corresponds to a measured lamination.

Conversely, suppose that  $v_1, \dots, v_\zeta$  corresponds to a measured lamination. If  $v_a + v_b > v_c$  then  $v_a + v_b - v_c > 10^{-(3h_1+3\zeta)}$  and so

$$x_a + x_b - x_c > 10^{-(3h_1+3\zeta)} - 10^{-(d_1-1)} > 10^{-p_1}.$$

Hence  $x_a + x_b > x_c$  to at least  $p_1$  decimal places. Similarly if  $v_a + v_b = v_c$  then  $x_a + x_b = x_c$  to at least  $p_1$  decimal places and so Stages 4–5 of the main algorithm will complete.  $\square$

From now on fix  $\mathcal{L}$  to be this measured lamination.

**Proposition 4.5.3.** *Stage 6 of the main algorithm completes if and only if  $\mathcal{L}$  is unitary (with respect to  $\mathcal{T}$ ).*

*Proof.* If  $\sum x_i = 1$  to  $p_1$  decimal places then  $\sum v_i - 1 = 0$  to at least  $p_1 - 1$  decimal places. However

$$\text{hgt}\left(\sum v_i - 1\right) + \text{dg}\left(\sum v_i - 1\right) \leq \zeta h_1 + \zeta^2 \leq p_1 - 1$$

and so again by Lemma 4.4.3 we have that  $\sum v_i = 1$ . Hence  $\mathcal{L}$  is unitary.

Conversely, if  $\mathcal{L}$  is unitary then  $\sum v_i - 1 = 0$  and so  $|\sum x_i - 1| \leq \zeta 10^{-d_1}$ . Hence  $\sum x_i = 1$  to at least  $p_1$  decimal places and so Stage 6 of the main algorithm will complete.  $\square$

Now following Stage 7, let  $B_i$  be such that  $B_i \cdot (x_1 \cdots x_\zeta)^T \geq 0$  and fix

$$(y_1 \cdots y_\zeta)^T := A_i \cdot (x_1 \cdots x_\zeta)^T \quad \text{and} \quad y := \sum y_i.$$

As  $v_i$  and  $x_i$  are so close and comparisons are only done to  $p_1$  decimal places we also have that  $B_i \cdot (v_1 \cdots v_\zeta)^T \geq 0$ . Hence we fix

$$(w_1 \cdots w_\zeta)^T := A_i \cdot (v_1 \cdots v_\zeta)^T \quad \text{and} \quad \lambda := \sum w_i.$$

Each  $A_i$  is  $\ell(p)$ -bounded and  $\ell(p) \leq h_1$ . Therefore each  $w_i$  has height at most  $h_2 := \zeta(2h_1 + 1)$  and agrees with  $y_i$  to at least  $d_2 := d_1 - \zeta h_1$  decimal places. Similarly,  $\lambda$  has height at most  $h_3 := \zeta(h_2 + 1)$  and agrees with  $y$  to at least  $d_3 := d_2 - \zeta$  decimal places.

**Proposition 4.5.4.** *Stage 8 of the main algorithm completes if and only if  $\mathcal{L}$  is projectively invariant under  $h$ , that is,  $h(\mathcal{L}) = \lambda \cdot \mathcal{L}$ .*

*Proof.* Firstly note that  $\lambda, y < 10^{h_3+\zeta}$  and so  $yx_i$  and  $\lambda v_i$  agree to at least  $d_3 - \zeta - h_3$  decimal places. Therefore, if  $y_i - yx_i = 0$  to  $p_1$  decimal places then  $w_i - \lambda v_i = 0$  to at least  $p_1 - 1$  places. However

$$\text{hgt}(w_i - \lambda v_i) + \text{dg}(w_i - \lambda v_i) \leq h_1 + h_2 + h_3 + 2\zeta^2 \leq p_1 - 1$$

and so again by Lemma 4.4.3 we have that  $w_i = \lambda v_i$ . Hence  $\mathcal{L}$  is projectively invariant under  $h$ .

Conversely, if  $\mathcal{L}$  is projectively invariant under  $h$  then  $w_i - \lambda v_i = 0$  and so

$$|y_i - yx_i| \leq 10^{-d_2} \leq 10^{-p_1} + 10^{-(d_3-\zeta-h_3)}.$$

Hence  $y_i = yx_i$  to at least  $p_1$  decimal places and so Stage 8 of the main algorithm will complete.  $\square$

Note that completing this stage shows that  $(v_1 \cdots v_\zeta)^T$  is an eigenvector of  $A_i$ . Hence each  $v_i$  lies in  $\mathbb{Q}(\lambda)$  and so any linear combination of them is also an algebraic number of degree at most  $\zeta$ .

**Proposition 4.5.5.** *Stage 9 of the main algorithm completes if and only if  $\mathcal{L}$  is stable, that is,  $\lambda > 1$ .*

*Proof.* If  $y - 1 > 0$  to  $p_1$  decimal places then  $\lambda - 1 > 0$  to at least  $p_1 - 1$  decimal places. However

$$\text{hgt}(\lambda - 1) + \text{dg}(\lambda - 1) \leq h_3 + \zeta \leq p_1 - 1.$$

and so again by Lemma 4.4.3 we have that  $\lambda > 1$ . Hence  $\mathcal{L}$  is stable.

Conversely if  $\mathcal{L}$  is stable then  $\lambda > 1 + 10^{-(h_3+\zeta)}$ . As  $\lambda$  and  $y$  agree to at least  $d_3$  decimal places we have that

$$y > 1 + 10^{-(h_3+\zeta)} - 10^{-d_3} > 1 + 10^{-p_1}.$$

Hence  $y > 1$  to at least  $p_1$  decimal places and so Stage 9 completes.  $\square$

Following Section 4.2, let  $T$  be the measured train track obtained from  $\mathcal{T}$  using  $v_1, \dots, v_\zeta$  and let  $T_i := s^i(T) = (\tau_i, \mu_i)$ . Similarly, let  $T'$  be the measured train track obtained from  $\mathcal{T}$  using  $x_1, \dots, x_\zeta$  instead and let  $T'_i := s^i(T') = (\tau'_i, \mu'_i)$ . We let  $v_i^k$  denote the weights on the branches of  $T_k$  and  $x_i^k$  denote the weights on the branches of  $T'_k$ .



**Proposition 4.5.6.** *Stage 10 of the main algorithm completes if and only if  $\mathcal{L}$  is filling.*

*Proof.* We show that  $T'_t$  is filling if and only if  $T_t$  is. The result then follows directly from Corollary 4.2.13.

We begin by claiming that for  $1 \leq k \leq t$ :

$$\tau_k = \tau'_k \quad \text{and} \quad |v_i^k - x_i^k| \leq 10^{-(d_1-k-1)}.$$

To see this first note that  $v_i^k$  is an algebraic number of degree at most  $\zeta$  and  $\text{hgt}(v_i^k) \leq 3\zeta(k + 2h_1 + 2)$  and that  $|v_i^0 - x_i^0| \leq 10^{-(d_1-1)}$ . Now suppose that  $\tau_k = \tau'_k$  and  $|v_i^k - x_i^k| \leq 10^{-(d_1-k-1)}$  for some  $1 \leq k < t$ . Then by Lemma 4.4.3,  $v_i^k \geq v_j^k$  if and only if  $x_i^k \geq x_j^k$  to  $p_1$  decimal places. Therefore the  $x_i^k$ -maximal branches are the  $v_i^k$ -maximal branches and so  $\tau_{k+1} = \tau'_{k+1}$ . Furthermore

$$|v_i^{k+1} - x_i^{k+1}| \leq |v_i^k - x_i^k| + |v_j^k - x_j^k| \leq 10^{-(d_1-k-2)}$$

and so the claim holds by induction on  $k$ .

Finally, again by Lemma 4.4.3, we have that  $v_i^t > 0$  if and only if  $x_i^t > 0$  to  $p_1$  decimal places. Hence  $T_t$  is filling if and only if  $T'_t$  is and so Stage 10 completes if and only if  $\mathcal{L}$  is filling.  $\square$

Combining these propositions we obtain:

**Corollary 4.5.7.** *Suppose that  $h \in \text{Mod}^+(S)$  is a mapping class and  $p$  is a path from  $\mathcal{T}$  to  $h(\mathcal{T})$ . If there is a certificate that the main algorithm accepts then  $h$  is pseudo-Anosov.*  $\square$

#### 4.5.2 Pseudo-Anosovs have acceptable certificates

Finally, we show the converse to Corollary 4.5.7. To do this we first require some additional bounds on the heights of certain algebraic numbers.

**Definition 4.5.8.** Suppose that  $\alpha \in \overline{\mathbb{Q}}$  is an algebraic number. A matrix  $M$  is  $\alpha$ -shifted if its entries are of the form  $a_{ij} = b_{ij} + c_{ij}\alpha$ , where  $b_{ij}$  and  $c_{ij}$  are integers. We say that such a matrix is  $k$ -bounded if each  $b_{ij}$  and  $c_{ij}$  is.

**Proposition 4.5.9.** *If  $M$  is a  $k$ -bounded,  $m \times m$ ,  $\alpha$ -shifted matrix then*

$$\text{hgt}(\det(M)) \leq m^2(k + \log(m) + \text{hgt}(\alpha)).$$

*Proof.* First note that we may expand  $(b_1 + c_1\alpha) \cdots (b_m + c_m\alpha)$  as a polynomial in  $\alpha$  to obtain  $d_0 + \cdots + d_m\alpha^m$ . It then follows from Fact 4.4.2 that  $\text{hgt}(d_i) \leq mk$  as  $b_i$  and  $c_i$  are  $k$ -bounded integers.

Now consider the following expansion of  $\det(M)$ :

$$\det(M) = \sum_{\sigma \in \text{Sym}(m)} \text{sign}(\sigma) \prod_{i=1}^m (b_{i\sigma(i)} + c_{i\sigma(i)}\alpha) = e_0 + \cdots + e_n\alpha^m.$$

By applying the previous bound to the coefficients of  $\prod_{i=1}^m (b_{i\sigma(i)} + c_{i\sigma(i)}\alpha)$  we have that

$$\text{hgt}(e_i) \leq mk + m \log(m).$$

Therefore:

$$\begin{aligned} \text{hgt}(\det(M)) &\leq \text{hgt}\left(\sum e_i \alpha^i\right) \\ &\leq \sum \text{hgt}(e_i \alpha^i) + m \log(2) \\ &\leq \sum \text{hgt}(e_i) + \frac{1}{2}m^2 \text{hgt}(\alpha) + m \log(2) \\ &\leq m^2k + m^2 \log(m) + \frac{1}{2}m^2 \text{hgt}(\alpha) + m \log(2) \\ &\leq m^2(k + \log(m) + \text{hgt}(\alpha)) \end{aligned}$$

Thus showing the required bound.  $\square$

**Lemma 4.5.10.** *Suppose that  $M$  is a  $k$ -bounded,  $m \times m$ ,  $\alpha$ -shifted matrix and that  $\det(M) \neq 0$ . If  $v = (\alpha_1 \cdots \alpha_m)^T$  is a vector of algebraic numbers such that  $M \cdot v$  is a  $k$ -bounded vector of integers then*

$$\text{hgt}(\alpha_i) \leq 2m^2(k + \log(m) + \text{hgt}(\alpha))$$

*Proof.* This bound follows from using Cramer's rule to determine  $\alpha_i$  and Proposition 4.5.9.  $\square$

**Lemma 4.5.11.** *Suppose that  $M$  is a  $k$ -bounded  $m \times m$  integer matrix. If  $\alpha$  is an eigenvalue of  $M$  then  $\text{hgt}(\alpha) \leq mk + m \log(m) + 2m$ .*

*Proof.* Again, following the proof of Proposition 4.5.9, we have that if  $\chi_M(x) = e_0 + \cdots + e_n x^m$  is the characteristic polynomial of  $M$  then  $\text{hgt}(e_i) \leq mk + m \log(m)$ . As  $\alpha$  is a root of this it follows from Fact 4.4.2 that

$$\text{hgt}(\alpha) \leq mk + m \log(m) + 2m. \quad \square$$

We now have the tools to prove existence of a certificate that the main algorithm will accept for pseudo-Anosovs.

**Theorem 4.5.12.** *Suppose that  $h \in \text{Mod}^+(S)$  is a mapping class and  $p$  is a path from  $\mathcal{T}$  to  $h(\mathcal{T})$ . If  $h$  is pseudo-Anosov then there is a certificate that the main algorithm will accept.*

*Proof.* Let  $\mathcal{L} = \mathcal{L}^+(h)$  be the stable lamination of  $h$  scaled such that  $\|\mathcal{T}(\mathcal{L})\| = 1$ . Let  $v_i := \mathcal{L}(e_i)$ , where  $e_i$  are the edges of  $\mathcal{T}$ , and  $\mathbf{v} := (v_1 \cdots v_\zeta)^T$ . Let  $\{A_i\}$  and  $\{B_i\}$  be the matrices of Lemma 3.3.2. There is  $B_i$  such that  $B_i \cdot \mathbf{v} \geq 0$  and so  $\mathbf{v}$  is an eigenvector of  $A_i$ . Therefore, as  $A_i$  is  $\ell(p)$ -bounded, each  $v_i$  is an algebraic number of degree at most  $\zeta$  and by Lemma 4.5.10 and Lemma 4.5.11 we have that

$$\begin{aligned} \text{hgt}(v_i) &\leq 2\zeta^2(\ell(p) + \log(\zeta) + \zeta\ell(p) + \zeta\log(\zeta) + 2\zeta) \\ &\leq 2\zeta^2((\zeta + 1)\ell(p) + (\zeta + 1)\log(\zeta) + 2\zeta) \\ &\leq \zeta^4(\ell(p) + 6) \\ &= h_0. \end{aligned}$$

Now let  $x_i$  be a decimal approximation of  $v_i$ , correct to  $d_1$  decimal places, and let  $f_i$  be the minimal integral polynomial of  $v_i$ . We claim that the certificate

$$x_1, \dots, x_\zeta, f_1, \dots, f_\zeta$$

is accepted by the main algorithm and proceed by considering each stage in turn:

- Stage 1. By definition each  $f_i$  has degree at most  $\zeta$  and height at most  $h_0$ . Hence this stage will pass.
- Stage 2. As  $0 \leq v_i \leq 1$  and  $|x_i - v_i| \leq 10^{-d_1}$ , we have that  $0 \leq x_i \leq 1$  to at least  $p_1$  decimal places. Hence this stage will pass.
- Stage 3. By Fact 4.4.2, two distinct roots of  $f_i$  must be separated by at least  $10^{-(2h_1+\zeta)}$  and as  $f_i$  is minimal it has no repeated roots. Hence,  $v_i$  is the unique root of  $f_i$  in  $[x_i - 10^{-d_1}, x_i + 10^{-d_1}]$  and so  $f_i(x_i \pm 10^{-d_1})$  must have different signs. Hence this stage will pass and the algebraic numbers found will be  $v_1, \dots, v_\zeta$ .

Finally, note that as  $v_1, \dots, v_\zeta$  represents  $\mathcal{L}$  which is unitary (with respect to  $\mathcal{T}$ ), projectively invariant, stable and filling Stages 6–10 must complete by Proposition 4.5.3, Proposition 4.5.4, Proposition 4.5.5 and Proposition 4.5.6. Proving the claim that the main algorithm will accept this certificate.  $\square$

Together with Corollary 4.5.7 this shows:

**Theorem 4.5.13.** *Suppose that  $h \in \text{Mod}^+(S)$  is a mapping class and  $p$  is a path from  $\mathcal{T}$  to  $h(\mathcal{T})$ . There is a certificate that the main algorithm will accept if and only if  $h$  is pseudo-Anosov.*  $\square$

**Remark 4.5.14.** By Proposition 4.5.6 and Lemma 4.2.8, there is also a certificate such that Stages 1 – 9 of the main algorithm will complete and Stage 10 will fail if and only if  $h$  is reducible.

## 4.6 Analysis

We analyse each of the stages of the main algorithm in turn. Here we use the natural extension of our initial model of computation to decimals with finitely many places. However this makes little difference as we only ever manipulate decimals with at most  $d_1 \in O(\ell(p)^2)$  places.

Recall that

$$h_0, h_1 \in O(\ell(p)) \quad \text{and} \quad t, d_1, p_1 \in O(\ell(p)^2).$$

Stage 1. This can be done in  $O(h_0) = O(\ell(p))$  operations.

Stage 2. This can be done in  $O(p_1) = O(\ell(p)^2)$  operations.

Stage 3. By expanding  $f_i(x) = \sum a_j x^j$  as  $a_0 + x(\cdots(a_{\zeta-2} + x(a_{\zeta-1} + a_{\zeta}x))\cdots)$  using Horner's rule [53, Section 4.6.4] it can be seen that each  $f_i(x_i \pm 10^{-d_1})$  can be computed in  $O(d_1(d_1 + \zeta h_0)) = O(\ell(p)^4)$ . Hence this stage can be done in  $O(\ell(p)^4)$  operations.

Stage 4. There are  $O(1)$  inequalities to check each of which requires  $O(d_1 + p_1) = O(\ell(p)^2)$  operations. Hence this can be done in  $O(\ell(p)^2)$  operations.

Stage 5. By the same argument this also requires  $O(\ell(p)^2)$  operations.

Stage 6. This can be done in  $O(d_1 + p_1) = O(\ell(p)^2)$  operations.

Stage 7. Following the ideas of Lemma 2.2.2, as each  $x_i$  has  $d_1$  digits, we can find  $A_i$  and  $B_i$  and compute  $y_1, \dots, y_{\zeta}$  in  $d_1 \ell(p) + \ell(p)^2 \in O(\ell(p)^3)$  operations. Each  $y_i$  has at most  $d_1 + \ell(p) = O(\ell(p)^2)$  digits. Hence,  $y$  can be computed in  $O(\ell(p)^2)$  further operations and has  $O(\ell(p)^2)$  digits. Therefore this entire stage can be done in  $O(\ell(p)^3)$  operations.

Stage 8. As  $x_i$ ,  $y_i$  and  $y$  each have  $O(\ell(p)^2)$  digits this stage can be done in  $O(\ell(p)^4 + d_1 + p_1) = O(\ell(p)^4)$  operations.

Stage 9. As  $y$  has  $O(\ell(p)^2)$  digits, this can be done in  $O(\ell(p)^2 + p_1) = O(\ell(p)^2)$  operations.

Stage 10. Constructing  $T'$  takes at most  $O(d_1) = O(\ell(p)^2)$  operations. For each maximal splitting it takes  $O(p_1)$  operations to find the maximal weight branches and then  $O(d_1)$  further operations to perform the splitting. Therefore we can construct  $T'_t := s^t(T')$  in  $O(t(p_1 + d_1)) = O(\ell(p)^4)$  operations. Finally, checking whether  $T'_t$  is filling can be done in  $O(p_1) = O(\ell(p)^2)$  operations. Hence this whole stage can be done in  $O(\ell(p)^4)$  time.

Therefore the main algorithm will terminate in  $O(\ell(p)^4)$  time.

Applying this result to the standard path of a word gives that:

**Corollary 4.6.1.** *Fix  $S$ , a marked surface, and  $X$ , a finite generating set of  $\text{Mod}^+(S)$ . Deciding whether a word in  $X^*$  is pseudo-Anosov is a problem in **NP**.  $\square$*

## Chapter 5

# The conjugacy problem

In this chapter we look at the conjugacy problem for mapping class groups. For ease of notation we will assume that  $S$  is connected, although it is straightforward to extend these results to the disconnected case. The pseudo-Anosov case covered in this chapter is also mentioned in [9, Section 6.3].

The conjugacy problem was studied by Dehn and described as one of the fundamental decision problems in group theory [32, Problem 2] [34]. For arbitrary groups this problem is undecidable [2] [78].

**Problem 5.0.1** (The conjugacy problem). Given words  $g, h \in X^*$  decide whether  $g$  and  $h$  represent *conjugate* mapping classes, that is, decide whether there is a word  $f \in X^*$  such that  $fgf^{-1} \equiv h$ .

The conjugacy problem for mapping class groups is also closely related to the homeomorphism problem for fibred 3-manifolds. The mapping tori  $M_g$  and  $M_h$  are fibrewise homeomorphic if and only if  $g$  is conjugate to  $h$  or  $h^{-1}$  [22, Proposition 5.11]. It was through this that Hemion first showed that the conjugacy problem is decidable [49]. However his solution is still not even known to be exponentially bounded [36, Page 24]. A partial solution in the pseudo-Anosov case was given by Mosher [66].

Most recently, Tao showed that mapping class groups have the Linearly Bounded Conjugator Property [84, Theorem B]. That is, there is a constant  $K = K(S, X)$  such that if two words  $g, h \in X^*$  represent conjugate mapping classes then there is a word  $f \in X^*$  such that

$$fgf^{-1} \equiv h \quad \text{and} \quad \ell(f) \leq K \cdot (\ell(g) + \ell(h)).$$

This establishes an exponential time solution but moreover as such a word  $f$  is sufficiently small it acts as a polynomial-time verifiable certificate that  $g$  and  $h$  are conjugate. This shows that:

**Theorem 5.0.2.** *Deciding whether two words  $g, h \in X^*$  correspond to conjugate mapping classes is a problem in  $\mathbf{NP}$ .*  $\square$

In this chapter we strengthen this theorem to include the complementary result:

**Corollary 5.5.6.** *Fix  $S$ , a marked surface, and  $X$ , a finite generating set of  $\text{Mod}^+(S)$ . Deciding whether two words  $g, h \in X^*$  correspond to conjugate mapping classes is a problem in  $\mathbf{NP} \cap \mathbf{co-NP}$ .*  $\square$

To do this we work with a path  $p$  from  $\mathcal{T}$  to  $h(\mathcal{T})$  and deal with four cases; depending on whether  $h$  is periodic or aperiodic and is reducible or irreducible. For each we describe a total conjugacy invariant and show that this can be constructed in  $O(\text{poly}(\ell(p)))$  time when given a suitable certificate.

Thus, if  $q$  is a second path from  $\mathcal{T}'$  to  $g(\mathcal{T}')$  then these invariants are small enough they can be compared in  $O(\text{poly}(\ell(p) + \ell(q)))$  operations. Hence we can determine whether  $g$  and  $h$  are conjugate in polynomial time. Applying this to the standard paths of  $g$  and  $h$  shows that there are certificates that allow us to determine whether  $g$  and  $h$  are conjugate in  $O(\text{poly}(\ell(g) + \ell(h)))$  operations.

In order to deal with the aperiodic, reducible case we will actually need to prove a stronger result. Namely, a solution to the *permutation conjugation problem*:

**Problem 5.0.3** (The permutation conjugacy problem). Suppose that  $g, h \in \text{Mod}^+(S)$  are mapping classes. Given paths  $p$  from  $\mathcal{T}$  to  $h(\mathcal{T})$  and  $q$  from  $\mathcal{T}$  to  $g(\mathcal{T}')$  and a map  $\pi: V \subseteq V(\mathcal{T}) \rightarrow V(\mathcal{T}')$ , decide whether  $g$  and  $h$  are  $\pi$ -conjugate. That is, decide whether there is a mapping class  $f \in \text{Mod}^+(S)$  such that  $f g f^{-1} \equiv h$  and  $f|_V = \pi$ .

## 5.1 Periodic mapping classes

We begin with the case in which  $h$  is periodic. Here we use properties of its quotient orbifold as a total conjugacy invariant.

As  $h$  is periodic, by the Nielsen realization theorem [37, Theorem 7.2] there is a homeomorphism  $\phi \in h$  such that  $\text{ord}(\phi) = \text{ord}(h)$ . We may use this to define the *quotient orbifold*  $\mathcal{O} := S / \phi$  which, up to homeomorphism, is independent of the particular choice of  $\phi$ .

There are two key properties of points of  $\mathcal{O}$  that we will need. Firstly, we say that the *order* of a point  $x \in \mathcal{O}$  is

$$\text{ord}(x) := \begin{cases} \infty & \text{if } x \text{ comes from the orbit of a marked point} \\ \text{ord}(\phi) / |\phi^{-1}(x)| & \text{otherwise.} \end{cases}$$

We note that all but finitely many of the points of  $\mathcal{O}$  are *regular*, that is, have order one, and in fact the number of irregular points is bounded in terms of the topology of  $S$ . Secondly, we note that even though  $\phi^{|\phi^{-1}(x)|}(x) = x$  this map may not act like the identity near  $x$  as it may rotate the tangent plane at  $x$  if  $x$  is irregular. Thus we define the *rotation number*  $r(x)$  of a point  $x \in \mathcal{O}$  to be the rational number in  $[0, 1)$  such that  $\phi^{|\phi^{-1}(x)|}$  rotates the tangent plane at  $x$  through  $2\pi r(x)$ . We note that the denominator of  $r(x)$  is bounded above by  $\text{ord}(\phi) \leq 8g(S) + 4n(S) - 2$  [37, Theorem 7.5].

Nielsen showed that this information at the irregular points

$$N := \{(\text{ord}(x), r(x)) : x \in \mathcal{O} \text{ and } \text{ord}(x) > 1\},$$

together with  $\text{ord}(h)$ , is a total conjugacy invariant of  $h$  [70, Theorem 9]. As periodic mapping classes have order at most  $8g(S) + 4n(S) - 2$  [37, Theorem 7.5] and there is a polynomial time solution to the word problem for mapping class groups [68], it only remains to show how to compute  $N$ . To do this computation we will divide into two subcases; depending on whether  $h$  is periodic and irreducible or periodic and reducible. Mosher showed that there is an algorithm that decides whether  $h$  is periodic in  $O(\text{poly}(\ell(h)))$  time [68]. We start by giving an algorithm to determine whether such an  $h$  is also irreducible.

We write  $H_1(\mathcal{O})$  for the integral first homology group of  $\mathcal{O}$  and, following our earlier notation, we write  $n(\mathcal{O})$  for the number marked points of  $\mathcal{O}$ , this is the number of orbits of marked points of  $S$  under  $h$ . It will also be convenient to define

$$\begin{aligned} \mathfrak{p} &:= |\text{Tor}(H_1(\mathcal{O}))| = \prod_{\substack{x \in \mathcal{O} \\ 1 < \text{ord}(x) < \infty}} \text{ord}(x), \\ \chi(\mathcal{O}) &= \chi(S) / \text{ord}(h), \text{ and} \\ g(\mathcal{O}) &= \frac{1}{2}(\text{rank}(H_1(\mathcal{O})) - n(\mathcal{O}) + 1). \end{aligned}$$

**Lemma 5.1.1.** *The periodic mapping class  $h$  is irreducible if and only if:*

1.  $g(\mathcal{O}) = 0$ ,



2.  $\chi = \chi(\mathcal{O}) \geq -1$  and either:
3. (a)  $n(\mathcal{O}) = 1$  and  $\mathfrak{p}^2(\chi + 1)^2 - 4\mathfrak{p}$  is a square,  
 (b)  $n(\mathcal{O}) = 2$  and  $\mathfrak{p}^2(\chi + 1)^2 - 4\mathfrak{p} \neq -16$ , or  
 (c)  $n(\mathcal{O}) = 3$ .

*Proof.* Start by noting that  $h$  is periodic and irreducible if and only if  $\mathcal{O}$  is a *triangle orbifold*, that is,  $\mathcal{O}$  has exactly three irregular points and its genus is 0. Moreover, as  $S$  is a marked surface, at least one of these points must be of infinite order.

In the forwards direction, suppose that  $\mathcal{O}$  is a triangle orbifold and so has signature  $(0; a, b, \infty)$  where  $a, b \in \{\infty, 2, 3, \dots\}$ . Properties 1 and 2 clearly hold as

$$\chi = -1 + \frac{1}{a} + \frac{1}{b} \geq -1.$$

Now for Property 3, if  $n(\mathcal{O}) = 1$  then  $\mathfrak{p} = ab$  and so

$$a^2 - (\chi + 1)\mathfrak{p}a + \mathfrak{p} = 0.$$

Hence

$$\mathfrak{p}^2(\chi + 1)^2 - 4\mathfrak{p} = (2a - (\chi + 1)\mathfrak{p})^2,$$

however this is the square of an integer as  $\chi\mathfrak{p}$  is integral. If  $n(\mathcal{O}) = 2$  then without loss of generality assume that  $a \neq \infty$  and  $b = \infty$ . Then

$$\chi = -1 + \frac{1}{a} \quad \text{and} \quad \mathfrak{p} = a.$$

and so  $\mathfrak{p}^2(\chi + 1)^2 - 4\mathfrak{p} = 1 - 4a \neq -16$ . Finally, if  $n(\mathcal{O}) = 3$  then there is nothing to check.

Conversely, if Properties 1 and 2 hold but  $\mathcal{O}$  is not a triangle orbifold then linear algebra shows that it must be one of the ones shown in Table 5.1. In any case, Property 3 cannot hold.  $\square$

We now give a polynomial time algorithm for computing  $H_1(\mathcal{O})$ . To ease this argument we set:

- $k := \text{ord}(\phi)$ ,
- $\pi_1(S) = \langle x_1, \dots, x_n \rangle$ ,
- $\phi_{\sharp}$  to be the induced action of  $\phi$  on  $\pi_1(S)$ , and
- $\phi_*$  to be the induced action of  $\phi$  on  $H_1(S)$ .

| Signature  | $\chi$             | $\mathfrak{p}$ | $\mathfrak{p}^2(\chi + 1)^2 - 4\mathfrak{p}$ |
|--|--------------------|----------------|--|
| $(0; 2, 2, a, \infty)$ where $2 \leq a < \infty$ | $-1 + \frac{1}{a}$ | $4a$           | $-16(a - 1)$                                 |
| $(0; 2, 3, 3, \infty)$                           | $-\frac{5}{6}$     | $18$           | $-63$  |
| $(0; 2, 3, 4, \infty)$                           | $-\frac{11}{12}$   | $24$           | $-92$  |
| $(0; 2, 3, 5, \infty)$                           | $-\frac{29}{30}$   | $30$           | $-119$                                       |
| $(0; 2, 3, 6, \infty)$                           | $-1$               | $36$           | $-144$                                       |
| $(0; 2, 4, 4, \infty)$                           | $-1$               | $32$           | $-128$                                       |
| $(0; 3, 3, 3, \infty)$                           | $-1$               | $27$           | $-108$                                       |
| $(0; 2, 2, 2, 2, \infty)$                        | $-1$               | $16$           | $-64$  |
| $(0; 2, 2, \infty, \infty)$                      | $-1$               | $4$            | $-16$  |

Table 5.1: Exceptional orbifolds which are almost triangle orbifolds.

Now note that the sequence

$$1 \rightarrow \pi_1(S) \rightarrow \pi_1(\mathcal{O}) \rightarrow \langle \phi \rangle \rightarrow 1$$

is exact [1, Page 44]. Hence there is a  $t \in \pi_1(\mathcal{O})$  such that  $t \mapsto \phi$  and so there is a  $w \in \pi_1(S)$  such that  $w \mapsto t^k$ . Therefore

$$\pi_1(\mathcal{O}) = \langle x_1, \dots, x_n, t \mid tx_it^{-1} = \phi_{\#}(x_i), t^k = w \rangle,$$

[17, Section 1] and so, as  $H_1(\mathcal{O})$  is the Abelianisation of  $\pi_1(\mathcal{O})$ , we have that

$$H_1(\mathcal{O}) = \langle x_1, \dots, x_n, t \mid x_i = \phi_*(x_i), t^k = [w], x_ix_j = x_jx_i, x_it = tx_i \rangle$$

where  $[w]$  is the homology class of the loop  $w$  chosen above. This group is equivalent to

$$\text{coker}(h_* - \text{Id}) \times \langle t \rangle / \langle t^k = [w] \rangle$$

and so, as the action of  $h_*$  with respect to some basis can be computed in polynomial time, computing  $H_1(\mathcal{O})$  reduces to computing  $[w]$ .

To compute the homology class  $[w]$ , we consider a *multiarc*, that is, is the isotopy class of the image of a smooth proper embedding of a finite number of copies of  $S^1$  and  $[0, 1]$  (whose endpoints are sent to marked points) into  $S$ . This non-standard definition of multiarc, in which components are also allowed to be closed loops, greatly simplifies the following argument. Again we represent a multiarc via its intersection numbers with the edges of a triangulation however, as a non-trivial multiarc can have zero intersection with all edges, we must first make a slight modification to our definition of intersection number.

**Definition 5.1.2.** If  $\alpha$  is a multiarc and the edge  $e$  of  $\mathcal{T}$  appears as a component of  $\alpha$  then their *intersection number*  $\iota(\alpha, e)$  is defined to be minus the number of copies of  $e$  that appear in  $\alpha$ .

Having made this change we can now identify a multiarc with its *normal coordinate*, its vector of intersection numbers with the edges of  $\mathcal{T}$ . These coordinates, allow us to restate many of the results of Chapter 2 and Chapter 3 for multiarcs.

Firstly, as in Lemma 3.3.1, there are  $O(1)$ -bounded  $\zeta \times 3\zeta$  matrices  $\overline{F}_i$  such that a vector  $v$  corresponds to a multiarc if and only if  $v \neq 0$  and

$$\overline{F}_i \cdot v \geq 0$$

for some  $i$ . Secondly, analysis of the 30 cases that can occur within a pair of triangles shows how these coordinates change under an edge flip.

**Lemma 5.1.3.** Suppose that  $\alpha$  is a multiarc and  $e$  is a flippable edge of a triangulation  $\mathcal{T}$  as shown in Figure 2.1 then

$$\iota(\alpha, e') = \begin{cases} \widehat{a} + \widehat{b} - \bar{e} & \text{if } \bar{e} \geq \widehat{a} + \widehat{b} \text{ and } \widehat{a} \geq \widehat{d} \text{ and } \widehat{b} \geq \widehat{c}, \\ \widehat{c} + \widehat{d} - \bar{e} & \text{if } \bar{e} \geq \widehat{c} + \widehat{d} \text{ and } \widehat{d} \geq \widehat{a} \text{ and } \widehat{c} \geq \widehat{b}, \\ \widehat{a} + \widehat{d} - \bar{e} & \text{if } \bar{e} \leq 0 \text{ and } \widehat{a} \geq \widehat{b} \text{ and } \widehat{d} \geq \widehat{c}, \\ \widehat{b} + \widehat{c} - \bar{e} & \text{if } \bar{e} \leq 0 \text{ and } \widehat{b} \geq \widehat{a} \text{ and } \widehat{c} \geq \widehat{d}, \\ \widehat{a} + \widehat{d} - 2\bar{e} & \text{if } \bar{e} \geq 0 \text{ and } \widehat{a} \geq \widehat{b} + \bar{e} \text{ and } \widehat{d} \geq \widehat{c} + \bar{e}, \\ \widehat{b} + \widehat{c} - 2\bar{e} & \text{if } \bar{e} \geq 0 \text{ and } \widehat{b} \geq \widehat{a} + \bar{e} \text{ and } \widehat{c} \geq \widehat{d} + \bar{e}, \\ \frac{1}{2}(\widehat{a} + \widehat{b} - \bar{e}) & \text{if } \widehat{a} + \widehat{b} \geq \bar{e} \text{ and } \widehat{b} + \bar{e} \geq 2\widehat{c} + \widehat{a} \text{ and } \widehat{a} + \bar{e} \geq 2\widehat{d} + \widehat{b}, \\ \frac{1}{2}(\widehat{c} + \widehat{d} - \bar{e}) & \text{if } \widehat{c} + \widehat{d} \geq \bar{e} \text{ and } \widehat{d} + \bar{e} \geq 2\widehat{a} + \widehat{c} \text{ and } \widehat{c} + \bar{e} \geq 2\widehat{b} + \widehat{d}, \\ \max(\widehat{a} + \widehat{c}, \widehat{b} + \widehat{d}) - \bar{e} & \text{otherwise.} \end{cases}$$

where  $\widehat{x} := \max(\iota(\alpha, x), 0)$  and  $\bar{e} := \iota(\alpha, e)$ . □

Now let  $v$  be a marked point of  $S$  and  $\alpha$  be an  $O(1)$ -bounded arc connecting from  $v$  to  $h(v)$ . By consistently pushing  $\alpha \cup \dots \cup h^{k-1}(\alpha)$  off of the marked points we obtain a loop  $\gamma$  which corresponds to  $w$  and is  $O(\ell(h))$ -bounded. From this we can determine  $[w]$  and so  $H_1(\mathcal{O})$  in polynomial time by again using the techniques of Erickson–Nayyeri [35, Section 6.6].

This, together with Lemma 5.1.1, gives an algorithm that determines whether  $h$  is periodic and irreducible in  $O(\text{poly}(\ell(p)))$  operations.

## 5.2 Periodic irreducible mapping classes

To compute  $N$  in the case in which  $h$  is periodic and irreducible we note that, even though  $\mathcal{O}$  does not contain a multicurve, it does contain a multiarc, as shown in Figure 5.1a. By lifting this back to  $S$  we see that there is an  $h$ -invariant multiarc  $\alpha$  on  $S$ . As  $h$  is irreducible,  $\alpha$  cannot contain any closed components and so  $\alpha$  decomposes  $S$  into polygonal pieces, each of which contains at most one preimage of an irregular point, as shown in Figure 5.1b. These polygons are permuted by the action of  $h$  and it is from this permutation that we can immediately determine the orders and rotation numbers of the irregular points of  $\mathcal{O}$  (including the points of infinite order) and so compute  $N$ .

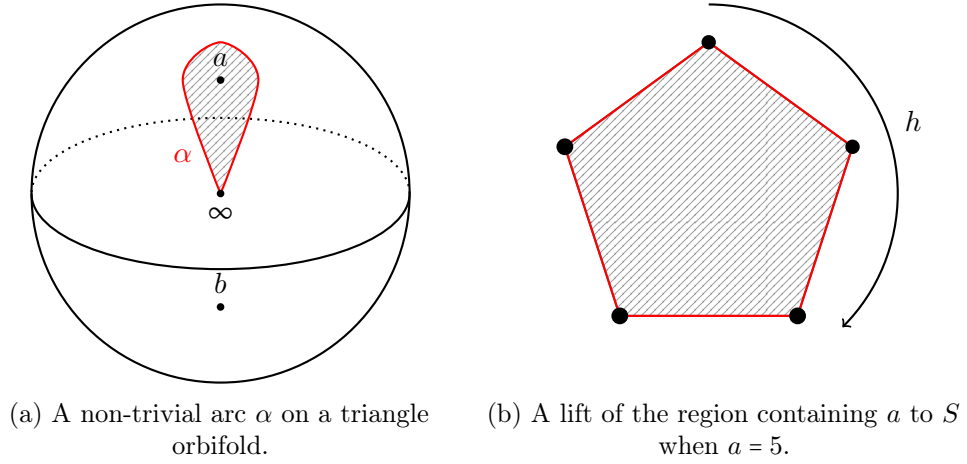


Figure 5.1: Lifting an arc from a triangle orbifold.

Thus given a path  $p$  from  $\mathcal{T}$  to  $h(\mathcal{T})$  and vector  $v$ , we use the following procedure to verify  $N$ :

- Stage 1. Check that the given vector corresponds to a multiarc  $\alpha$  by checking that  $\overline{F}_i \cdot v \geq 2$  for some  $i$ .
- Stage 2. Check that  $\alpha$  is  $h$ -invariant by computing and checking that  $\mathcal{T}(h(\alpha)) = v$ .
- Stage 3. Compute the normal coordinate and multiplicity of each component  $\alpha_i$  of  $\alpha$ .
- Stage 4. Compute the components  $\alpha_i$  that are adjacent to each region of  $S - \alpha$ , together with their cyclic ordering.
- Stage 5. Compute the permutation of  $\alpha_i$  induced by  $h$  and so compute  $N$ .

To analyse the number of operations required by this procedure, suppose that the given vector  $v$  is  $k$ -bounded. Then checking that  $\overline{F}_i \cdot v \geq 2$  for some  $i$

can be done in  $O(\text{poly}(k))$  operations as the  $\overline{F}_i$  matrices are  $O(1)$ -bounded. By repeating the idea of Lemma 2.2.2, we can compute  $\mathcal{T}(h(\alpha))$  and so complete Stage 2 in  $O(\text{poly}(k) \text{poly}(\ell(p)))$  operations. We can then use the algorithm of Agol, Hass and Thurston [5] to complete Stage 3 and Stage 4 in  $O(\text{poly}(k))$  operations. Finally by computing  $h(\alpha_i)$  using the idea of Lemma 2.2.2, this permutation can be computed in  $O(\text{poly}(k) \text{poly}(\ell(p)))$  operations and from this  $N$  can be computed in  $O(\text{poly}(k) \text{poly}(\ell(p)))$  operations.

Now note that if we are given a path  $p$  from  $\mathcal{T}$  to  $h(\mathcal{T})$  we can once again compute matrices  $\overline{A}_i$  and  $\overline{B}_i$  which describe the piecewise linear transformation between coordinates with respect to  $\mathcal{T}$  and  $h(\mathcal{T})$ . Again, these matrices can be chosen such that:

- each  $\overline{A}_i$  and  $\overline{B}_i$  is  $\ell(p)$ -bounded,
- each  $\overline{B}_i$  has  $O(\ell(p))$  rows,
- for each multiarc  $\alpha$  we have that  $\overline{B}_i \cdot \mathcal{T}(\alpha) \geq 0$  for some  $i$ , and
- for each multicurve  $\alpha$  we have that

$$\mathcal{T}(h(\alpha)) = \overline{A}_i \cdot \mathcal{T}(\alpha) \quad \text{if and only if} \quad \overline{B}_i \cdot \mathcal{T}(\alpha) \geq 0.$$

Thus, by repeating the argument of Theorem 3.3.3 using  $\overline{A}_i$ ,  $\overline{B}_i$  and  $\overline{F}_i$  we deduce that there is a multiarc  $\alpha$  on  $S$  such that  $h(\alpha) = \alpha$  and  $\mathcal{T}(\alpha)$  is  $O(\ell(p))$ -bounded. Hence there is a  $v$  such that we can compute  $N$  in  $O(\text{poly}(\ell(p)))$  operations.

**Corollary 5.2.1.** *Deciding whether two periodic, irreducible words  $g, h \in X^*$  correspond to conjugate mapping classes is a problem in  $\mathbf{NP} \cap \mathbf{co-NP}$ .*  $\square$

### 5.3 Periodic reducible mapping classes

In the case in which  $h$  is periodic and reducible we can still compute  $g(\mathcal{O})$ ,  $n(\mathcal{O})$ ,  $\chi(\mathcal{O})$  and  $\mathfrak{p}$  in polynomial time as before. To find  $N$  we use an  $h$ -maximal multicurve  $\gamma$ , which we may assume to be  $O(\ell(p))$ -bounded by Theorem 3.4.8. We crush  $S$  along  $\gamma$  and examine each piece of the induced mapping class  $h_\gamma \in \text{Mod}^+(S_\gamma)$  in turn. Suppose that  $S' \subseteq S_\gamma$  is obtained by taking the orbit of one component of  $S_\gamma$  under  $h_\gamma$ . Let  $h' \in \text{Mod}^+(S')$  be the mapping class induced by  $h_\gamma$  and  $\mathcal{T}'$  the triangulation of  $S'$  induced by  $\mathcal{T}_\gamma$ . Using the results of Section 3.4, we can construct a path  $p'$  from  $\mathcal{T}'$  to  $h'(\mathcal{T}')$  such that  $\ell(p') \leq \ell(p)$ . The previous section allows us to compute the signature of the orbifold  $S'/h'$  by using  $p'$ . Now if  $x \in S'/h'$  and  $\text{ord}(x) < \infty$  then  $x$

lifts to a point  $y \in S/h$  and  $\text{ord}(y) = \text{ord}(x)$ . Repeating this for all possible  $S'$  allows us to compute the orders of many of the irregular points of  $\mathcal{O}$ .

To deal with any omitted points, we note that there is also a representative  $c \in \gamma$  such that  $\phi(c) = c$ . We may perform the previous procedure using  $c$ ; crushing  $S$  along  $c$  to obtain  $S_c$  with induced homeomorphism  $\phi_c$ . Now if  $y$  is an irregular point of  $\mathcal{O}$  which is omitted by the previous construction then  $y$  must either:

- lie on one of the components of  $c$ , or
- be contained in an annular region, bounded by two of the components of  $c$ .

Regardless, the order of  $y$  must be 2. Therefore by comparing the product of the list of orbifold point orders found and  $\mathfrak{p}$  we can determine exactly how many of points of order two we are missing. Finally, as  $r(x) \neq 0$ , the rotation numbers of these points must be  $\frac{1}{2}$  and so we can compute  $N$ .

To bound the complexity of this construction we use the results of Section 3.4. We can compute  $\mathcal{T}_\gamma$  and  $p_\gamma$  in  $O(\text{poly}(\ell(p)))$  operations by Theorem 3.4.2. From this for each choice of  $S' \subseteq S_\gamma$  we can construct  $p'$  in  $O(\text{poly}(\ell(p)))$  operations. Now as each such path has length at most  $\ell(p)$  we can compute the quotient orbifold  $S'/h_\gamma$  in  $O(\text{poly}(\ell(p)))$  operations by the previous section. Finally determining how many orbifold points of order two we are missing by comparing  $\mathfrak{p}$  and the product of the list of orbifold point orders found can also be done in  $O(\text{poly}(\ell(p)))$  operations. All together this shows that we can compute  $N$  in  $O(\text{poly}(\ell(p)))$  operations.

Therefore, using the standard path for  $h \in X^*$  and given a  $O(\ell(h))$ -bounded  $h$ -maximal multicurve  $\gamma$  we see that this procedure takes polynomial time. From which we determine that:

**Lemma 5.3.1.** *Deciding whether two periodic, reducible words  $g, h \in X^*$  correspond to conjugate mapping classes is a problem in  $\mathbf{NP} \cap \mathbf{co-NP}$ .*  $\square$

Together with Corollary 5.2.1 this shows that:

**Corollary 5.3.2.** *Deciding whether two periodic words  $g, h \in X^*$  correspond to conjugate mapping classes is a problem in  $\mathbf{NP} \cap \mathbf{co-NP}$ .*  $\square$

For a point  $x \in S$  we define its rotation number (with respect to  $h$ ) to be the rotation number of its image in the  $S/h$  quotient orbifold.

**Lemma 5.3.3.** *Periodic mapping classes  $h$  and  $g$  are  $\pi$ -conjugate if and only if:*

- $h$  and  $g$  are conjugate, and

- for each marked point  $v$  we have that:

- $v$  and  $\pi(v)$  have the same rotation number (with respect to  $h$  and  $g$  respectively), and
- $\pi(h^k(v)) = g^k(\pi(v))$  for every  $k \in \mathbb{Z}$

whenever these maps are defined.

*Proof.* The forward direction of this lemma holds trivially. For the reverse direction, we first note that without loss of generality we may assume that  $\pi$  is actually a permutation of the vertices of  $S$ . If it is not then we consider each of the possible extensions of  $\pi$  in turn.

As  $g$  and  $h$  are conjugate there is a mapping class  $f$  such that  $h = f^{-1}gf$ . If  $f|_V \neq \pi$  then consider  $\varphi: S/h \rightarrow S/g$ , the homeomorphism induced by  $f$ . This homeomorphism respects the orders and rotation numbers of points and satisfies the *lifting criterion*: it maps the subgroup  $\pi_1(S) \leq \pi_1(S/h)$  to the subgroup  $\pi_1(S) \leq \pi_1(S/g)$ . We can modify  $\varphi$  by precomposing it with another homeomorphism  $\psi: S/h \rightarrow S/h$ . If we take  $\psi$  to be a homeomorphism that swaps two of the marked points with the same rotation number then  $\psi$  preserves the subgroup  $\pi_1(S) \leq \pi_1(S/h)$ . Hence  $\varphi \circ \psi$  also satisfies the lifting criterion and so lifts to an alternate mapping class  $f'$  which, like  $f$ , conjugates  $g$  to  $h$  but whose action on  $V$  is permuted by  $\psi$ .

Therefore, as  $\pi$  sends marked points to marked points with the same rotation number and  $\pi(h^k(v)) = g^k(\pi(v))$  for every  $k \in \mathbb{Z}$ , modifications of this form are sufficient to adjust  $f$  such that  $f|_V = \pi$ . Hence  $h$  and  $g$  are  $\pi$ -conjugate.  $\square$

The rotation numbers of the marked points can be determined in polynomial time from the polygonal decomposition of  $S$  given by the multiarc  $w$ . Thus this additional criterion can also be tested in polynomial time.

## 5.4 Aperiodic irreducible mapping classes

By the Nielsen–Thurston classification, a mapping class is aperiodic and irreducible if and only if it is pseudo-Anosov [37, Theorem 13.2]. Agol showed that for these mapping classes the combinatorics of the periodic part of their maximal splitting sequence is a total conjugacy invariant [3, Section 7]. Thus, as we can construct these in polynomial time when given an acceptable certificate for  $p$ , we immediately obtain that:

**Corollary 5.4.1.** *Deciding whether two aperiodic irreducible words  $g, h \in X^*$  correspond to conjugate mapping classes is a problem in  $NP \cap co-NP$ .*  $\square$

We also note that in this case  $g$  and  $h$  are  $\pi$ -conjugate if and only if they are conjugate and there is a map between the combinatorics of the periodic part of their maximal splitting sequence which induces  $\pi$  on  $V$ . Again, given acceptable certificates for  $p$  and  $q$ , this can be done in  $O(\text{poly}(\ell(p) + \ell(q)))$  operations.

## 5.5 Aperiodic reducible mapping classes

To deal with the remaining case when  $h$  is aperiodic and reducible we use its partition graph as a total conjugacy invariant.

As in Section 3.4.2, let  $\sigma(h) \neq \emptyset$  be the canonical curve system of  $h$  with components  $\{\gamma_j\}$ . Let  $S_{\sigma(h)}$  be the surface obtained by crushing  $S$  along  $\sigma(h)$  and for ease of notation let  $\{S_i\}$  be its connected components. Additionally, let  $h_i$  be the mapping class induced on  $S_i$  by the first return map of  $h$ , these mapping classes are the *canonical components* of  $h$ .

**Definition 5.5.1** ([61, Theorem 2], [70]). The *partition graph* of  $h$  is the pair  $(H, \phi)$  where:

- $H$  is the finite graph with:
  - a vertex corresponding to each  $h_i$ , and
  - an edge corresponding to each  $\gamma_i$ , connecting between  $h_j$  and  $h_k$  when  $\gamma_i$  meets  $S_j$  and  $S_k$ .
- $\phi$  is the automorphism of  $H$  induced by  $h$ .

For example, consider the collection of curves on  $S_{2,2}$  shown in Figure 5.2. The mapping class  $h = T_\alpha \circ T_\beta^{-1} \circ T_\gamma \circ T_\delta$  is aperiodic and reducible. Its partition graph is also shown in Figure 5.2 and has two edges corresponding to two components of  $\sigma(h)$ , the blue curves. The mapping class  $h_1 \in \text{Mod}^+(S_{1,2})$  is pseudo-Anosov while the mapping class  $h_2 \in \text{Mod}^+(S_{0,4})$  is the identity map. Furthermore, the automorphism of the partition graph induced by  $h$  is the identity map.

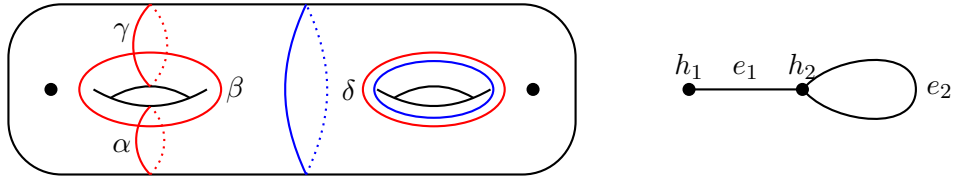


Figure 5.2: Curves on  $S_{2,2}$  and the partition graph for  $T_\alpha \circ T_\beta^{-1} \circ T_\gamma \circ T_\delta$ .



Most importantly, we can provide a polynomial-time verifiable certificate that a given graph is the partition graph of  $h$ . This certificate consists of two pieces of information. Firstly  $\mathcal{T}(\sigma(h))$  which is  $O(\ell(p))$ -bounded by Proposition 3.4.10 and so from which we can compute paths representing each  $h_i$  in  $O(\text{poly}(\ell(p)))$  operations by using the algorithm of Theorem 3.4.2. Secondly a certificate accepted by the main algorithm for each pseudo-Anosov  $h_i$ , this allows us to deduce that the given multicurve is  $h$ -maximal. To see that this multicurve is actually the canonical curve system, we note that it is sufficient to check that for any  $\gamma_i$ , removing its orbit under  $h$  from  $\sigma(h)$  does not result in a new periodic component. As there are at most  $\zeta \in O(1)$  such orbits to check and we can compute each  $\mathcal{T}(\gamma_i)$  in  $O(\text{poly}(\ell(p)))$  operations by the algorithm of Proposition 2.3.1, this can also be done in polynomial time.

### 5.5.1 Twist invariants

Associated to each component  $\gamma_i$  of the canonical curve system is its *twist invariant*  $d_i$ . This is a rational number describing the number of (fractional) Dehn twists performed about it [70]. We now explicitly describe how to compute these numbers given some additional multicurves.

**Definition 5.5.2.** A curve  $\gamma$  is *dual* to a component  $\gamma_i$  of  $\sigma(h)$  if

$$\iota(\gamma, \gamma_i) = \begin{cases} 2 & \text{if } \gamma_i \text{ is separating} \\ 1 & \text{otherwise.} \end{cases}$$

Now suppose that  $\gamma$  is dual to  $\gamma_i$  and let  $\delta := \partial N(\gamma_i \cup \gamma)$ . Let  $k := \text{lcm}(1, 2, \dots, 4\zeta)$  then

$$|d_i| = \frac{\iota(h^k(\gamma), \gamma) - \frac{1}{2} \iota(h^k(\gamma), \delta)}{k \iota(\gamma, \gamma_i)}.$$

This particular value of  $k$  was chosen such that  $h^k$  fixes all components of  $\sigma(h)$  and all prongs of singularities of any measured lamination that is projectively invariant under  $h$ .

To compute the sign of  $d_i$  we check whether the number of intersections grows when an additional Dehn twist along  $\gamma_i$  is performed. That is,  $d_i \geq 0$  if and only if

$$\iota(h^k(T_{\gamma_i}(\gamma)), \gamma) \geq \iota(h^k(\gamma), \gamma).$$

To analyse the complexity of computing these quantities we first require a technical lemma:

**Lemma 5.5.3.** *Suppose that  $\gamma, \gamma' \in \mathcal{C}(S)$  are multicurves and let  $\delta := \partial N(\gamma \cup \gamma') \in \mathcal{C}(S)$ , after possibly removing any null-homotopic components. If  $\mathcal{T}(\gamma)$  and  $\mathcal{T}(\gamma')$  are  $k$ -bounded then  $\mathcal{T}(\delta)$  is  $(k+1)$ -bounded.  $\square$*

Following this as  $\mathcal{T}(\gamma_i)$  is  $O(\ell(p))$ -bounded there is a dual curve  $\gamma$  such that  $\mathcal{T}(\gamma)$  is also  $O(\ell(p))$ -bounded and moreover by Lemma 5.5.3  $\mathcal{T}(\delta)$  is also  $O(\ell(p))$ -bounded. Therefore, by Proposition 2.3.3, given  $\mathcal{T}(\gamma)$  for such a  $\gamma$  we can verify that  $\gamma$  is dual to  $\gamma_i$ . Furthermore given  $\mathcal{T}(\delta)$  we can crush  $S$  along  $\delta$  and check that the component of  $S_\delta$  containing  $\gamma_i$  and  $\gamma$  is either a four times marked sphere or a once-marked torus, depending on whether  $\gamma_i$  is separating or not. This verifies that  $\delta = \partial N(\gamma_i \cup \gamma)$ . Having established that  $\gamma$  and  $\delta$  are the relevant curves, we can use  $\gamma$  and  $\delta$  to compute  $d_i$  in  $O(\text{poly}(\ell(p)))$  operations by the previous formulae and Proposition 2.3.3.

In the case of the example shown in Figure 5.2, we compute that the twist invariant of  $e_1$  is 0 while the twist invariant of  $e_2$  is +1.

### 5.5.2 Equivalence of partition graphs

Now suppose that  $(G, \phi)$  and  $(H, \psi)$  are the partition graphs of  $g$  and  $h$  respectively. A graph isomorphism  $\Phi: G \rightarrow H$  induces a bijection between the marked points of  $g_i$  coming from  $\sigma(g)$  and those of  $\Phi(g_i)$  coming from  $\sigma(h)$ . We denote this by  $D_{g_i}\Phi$ .

**Theorem 5.5.4** ([70, Theorem 8.3] [69, Section 1.14]). *Suppose that  $(G, \phi)$  and  $(H, \psi)$  are the partition graphs of  $g$  and  $h$  respectively. Then  $g$  and  $h$  are conjugate if and only if there is a graph isomorphism  $\Phi: G \rightarrow H$  such that:*

1.  $\Phi$  conjugates  $\phi$  to  $\psi$ ,
2.  $g_i$  and  $\Phi(g_i)$  are  $D_{g_i}\Phi$ -conjugate, and
3.  $\gamma_j$  and  $\Phi(\gamma_j)$  have the same twist invariant.  $\square$

Thus when  $g$  and  $h$  are not conjugate, for each graph isomorphism  $\Phi: G \rightarrow H$  either:

1.  $\Phi \circ \phi \neq \psi \circ \Phi$ , which we can check in  $\zeta \in O(1)$  operations,
2. there is a vertex  $g_i \in G$  such that  $g_i$  and  $\Phi(g_i)$  are not  $D_{g_i}\Phi$ -conjugate and so by the earlier sections of this chapter we can provide a polynomial-time checkable proof of this fact, or

3. there is an edge  $\gamma_j \in G$  such that  $\gamma_j$  and  $\Phi(\gamma_j)$  have different twist invariants and so by Lemma 5.5.1 we can provide a polynomial-time checkable proof of this fact.

By doing this for each of the at most  $\zeta! \in O(1)$  such isomorphisms, we can provide a proof that  $g$  and  $h$  are not conjugate which can be verified in  $O(\text{poly}(\ell(p) + \ell(q)))$  operations.

Again, applying this result to the standard path of a word we obtain that:

**Corollary 5.5.5.** *Deciding whether two aperiodic, reducible words  $g, h \in X^*$  correspond to conjugate mapping classes is a problem in  $\mathbf{NP} \cap \mathbf{co-NP}$ .*  $\square$

Together with Corollary 5.4.1 and Corollary 5.3.2 this shows that:

**Corollary 5.5.6.** *Fix  $S$ , a marked surface, and  $X$ , a finite generating set of  $\text{Mod}^+(S)$ . Deciding whether two words  $g, h \in X^*$  correspond to conjugate mapping classes is a problem in  $\mathbf{NP} \cap \mathbf{co-NP}$ .*  $\square$

Once more, this gives an exponential time algorithm to determine whether two words represent conjugate mapping classes.

**Remark 5.5.7.** This approach of constructing and comparing a total conjugacy invariant of  $h$  can be extended to partition a set of mapping classes  $\{h_1, \dots, h_k\}$  into conjugacy classes. In this case, by first computing all of the total conjugacy invariants and then comparing these in pairs we can do this in

$$k \exp(O(\max\{\ell(h_i)\})) + k^2 O(\text{poly}(\max\{\ell(h_i)\}))$$

operations. When  $k$  is large, this is a significant improvement over performing the naïve pairwise comparisons, which takes  $k^2 \exp(O(\max\{\ell(h_i)\}))$  operations.

## Chapter 6

# Other applications and implementation

In this chapter we discuss some additional consequences of Corollary 4.6.1 and Corollary 5.5.6. In particular further properties of  $h \in \text{Mod}^+(S)$  that can be efficiently determined from a train track on the axis of  $h$ , many of which follow from results mirroring Lemma 4.2.8. We go on to discuss some details of an implementation of the main algorithm, as part of the Python package `flipper`, and give several examples of the kinds of calculations that can be done using it. Finally we finish with some further extensions and conjectures.

### 6.1 Applications

Several of the problems that follow are not decision problems but are function problems, a closely related class in which more than a yes / no output is allowed.

**Definition 6.1.1** ([48, Appendix A.13]). A function  $f: A \rightarrow B$  is in **FNP** if there is a polynomial  $q \in \mathbb{Z}[x]$  and Turing machine  $M$  such that for each  $a \in A$ : there is a certificate  $c$  such that  $M$  accepts  $(a, b, c)$  in time  $q(|a|)$  if and only if  $f(a) = b$ .

This is the functional equivalent of **NP**.

#### 6.1.1 Nielsen–Thurston types

We have already seen several results regarding Nielsen–Thurston types. In summary, deciding whether a word is:

- periodic is a problem in  $\mathbf{P} \subseteq \mathbf{NP} \cap \mathbf{co-NP}$  [68].

- periodic and irreducible is a problem in  $\mathbf{P} \subseteq \mathbf{NP} \cap \mathbf{co-NP}$  (Section 5.1).
- reducible is a problem in  $\mathbf{NP}$  (Corollary 3.3.5).
- pseudo-Anosov is a problem in  $\mathbf{NP}$  (Corollary 4.6.1).

Combining these with the Nielsen–Thurston classification theorem (Theorem 1.0.1) we obtain that:

**Proposition 6.1.2.** *When  $S$  is connected, deciding whether  $h \in X^*$  is reducible is a problem in  $\mathbf{NP} \cap \mathbf{co-NP}$ .*  $\square$

**Proposition 6.1.3.** *When  $S$  is connected, deciding whether  $h \in X^*$  is pseudo-Anosov is a problem in  $\mathbf{NP} \cap \mathbf{co-NP}$ .*  $\square$

### 6.1.2 Dilatation

If  $h \in X^*$  is pseudo-Anosov then its *dilatation*  $\lambda^+(h)$  is the rescaling constant of its stable lamination. For completeness we define  $\lambda^+(h) := 1$  when  $h$  is not pseudo-Anosov [37, Section 13.2.3]. As we have seen in Section 4.5.1, the dilatation is an algebraic number and so we may represent it by a triple  $(d, h, x)$  where  $d$  and  $h$  are integers and  $x$  is a decimal such that:

$$d \geq \text{dg}(\alpha), \quad h \geq \text{hgt}(\alpha) \quad \text{and} \quad |x - \alpha| \leq 10^{-(d+h)}.$$

One algebraic number can be represented in several different ways under this scheme although it is straightforward to determine whether two such representations are of the same number.

Now when  $h \in X^*$  is pseudo-Anosov its dilatation is well approximated by the value  $y$ , obtained in Stage 7 of the main algorithm. This shows that:

**Theorem 6.1.4.** *The function  $\lambda^+: X^* \rightarrow \overline{\mathbb{Q}}$  taking a word to the dilatation of its corresponding mapping class is in  $\mathbf{FNP}$ .*  $\square$

### 6.1.3 Stratum

When  $h \in \text{Mod}^+(S)$  is a pseudo-Anosov mapping class, its stable lamination  $\mathcal{L} = \mathcal{L}^+(h)$  can be extended to a singular foliation  $\mathcal{F}$  on  $S$  without any saddle connections. This foliation has a product structure at all but finitely many points, where it has a singularity of order  $k$ . See Figure 6.1. The multiset of orders of the singularities of  $\mathcal{F}$  determines the *stratum* of  $h$ .

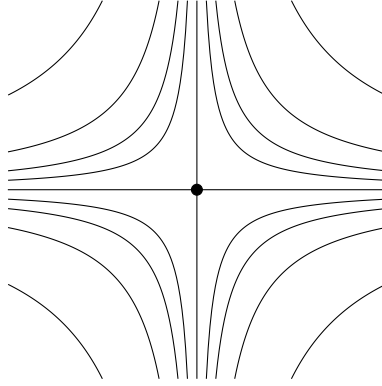


Figure 6.1: A singularity of order 4.

If  $\mathbf{T}_i = (\tau, \mu)$  is on the axis of  $\mathcal{L}$  then  $\mathcal{F}$  has one singularity for each component of  $S - \tau$ . Moreover the order of such a singularity is equal to the number of cusps of the corresponding component of  $S - \tau$ . Given an acceptable certificate for  $h$  we can construct  $\mathbf{T}_i = (\tau, \mu)$  in polynomial time and so compute the stratum of  $h$  in  $O(\text{poly}(\ell(h)))$  operations.

**Lemma 6.1.5.** *The function  $\text{strat}(h)$  taking a word to the multiset of singularity orders when  $h$  is pseudo-Anosov and the empty set otherwise is in **FNP**.  $\square$*

#### 6.1.4 Orientability

A pseudo-Anosov mapping class  $h \in \text{Mod}^+(S)$  defines a *quadratic differential* on  $S$  via its stable lamination  $\mathcal{L} = \mathcal{L}^+(h)$ . This quadratic differential is an *Abelian differential* if and only if  $\mathcal{L}$  is *orientable*, that is, the leaves of its underlying lamination can be assigned a locally consistent orientation.

**Definition 6.1.6.** A measured train track  $T = (\tau, \mu)$  is *orientable* if each branch of  $\tau$  can be assigned an orientation which agrees at the switches.

**Lemma 6.1.7.** *Suppose that  $\mathbf{T}_i$  is a train track on the axis of  $h$ . Then  $\mathcal{L}$  is orientable if and only if  $\mathbf{T}_i$  is.*

The proof follows that of Lemma 4.2.8.

*Proof.* If  $\mathcal{L}$  is orientable then there is a measured train track  $T = (\tau, \mu)$ , representing  $\mathcal{L}$ , which is orientable. By Theorem 4.2.5 there are  $j$  and  $k$  such that  $\widehat{h}^k(s^j(T)) = \mathbf{T}_i$ . As  $T$  is orientable and this is preserved by both maximal splittings and homeomorphisms,  $\mathbf{T}_i$  is orientable too.

In the reverse direction, if  $\mathbf{T}_i = (\tau, \mu)$  is orientable then its orientation can be pulled back to  $\mathcal{L}$  and so  $\mathcal{L}$  is orientable too.  $\square$

As we can construct a train track on the axis of  $h$  in  $O(\text{poly}(\ell(h)))$  operations, given an acceptable certificate for  $h$ , we obtain:

**Corollary 6.1.8.** *Deciding whether a pseudo-Anosov word in  $X^*$  corresponds to an Abelian differential on  $S$  is a problem in  $\mathbf{NP} \cap \mathbf{co-NP}$ .*  $\square$

### 6.1.5 Commuting

Suppose that  $h \in \text{Mod}^+(S)$  is a pseudo-Anosov mapping class and that  $\mathcal{L} = \mathcal{L}^+(h)$  is its stable lamination. A periodic mapping class  $g \in \text{Mod}^+(S)$  commutes with  $h$  if and only if  $g(\mathcal{L}) = \mathcal{L}$ .

**Lemma 6.1.9.** *Suppose that  $h \in \text{Mod}^+(S)$  is a pseudo-Anosov mapping class and  $\mathbf{T}_i$  is a train track on its axis. Any finite order mapping class  $g \in \text{Mod}^+(S)$  that commutes with  $h$  arises as an automorphism of  $\mathbf{T}_i$ .*  $\square$

One immediate consequence of this lemma is that there are at most  $3\zeta$  periodic mapping classes that commute with  $h$ . More importantly, given an acceptable certificate for  $h$ , we can compute  $\mathbf{T}_i$  and so all periodic mapping classes that commute with  $h$  in  $O(\text{poly}(\ell(h)))$  operations.

**Corollary 6.1.10.** *The function taking a word  $h \in X^*$  to:*

- *the set of periodic mapping class that commute with  $h$  if  $h$  is pseudo-Anosov, and*
- *the empty set otherwise*

*is in  $\mathbf{FNP}$ .*  $\square$

Hence there is an exponential time algorithm that, given a path  $p$  from  $\mathcal{T}$  to  $h(\mathcal{T})$ , computes the set of periodic mapping classes which fix  $\mathcal{L} = \mathcal{L}^+(h)$ . This answers Question 3.28 of [45].

### 6.1.6 Roots

**Definition 6.1.11.** Suppose that  $h \in \text{Mod}^+(S)$  is a mapping class. A mapping class  $g \in \text{Mod}^+(S)$  is an  $n^{\text{th}}$  root of  $h$  if  $g^n = h$ .

Again, we focus on the case when  $h \in \text{Mod}^+(S)$  is pseudo-Anosov. Here we note that if  $g \in \text{Mod}^+(S)$  is an  $n^{\text{th}}$  root of  $h$  then it is also pseudo-Anosov and moreover

$$\lambda^+(g) = \sqrt[n]{\lambda^+(h)}.$$

Therefore,  $h$  has only finitely many roots as

$$\{\lambda^+(g) : g \in \text{Mod}^+(S)\} \subseteq \mathbb{R}$$

is discrete [37, Lemma 12.4] and any two  $n^{\text{th}}$  roots of  $h$  differ by a periodic mapping class which fixes  $\mathcal{L}^+(h)$ .

**Lemma 6.1.12.** *Suppose that  $h \in \text{Mod}^+(S)$  is a pseudo-Anosov mapping class,  $\{\mathbf{T}_i\}_{i=-\infty}^{\infty}$  is its axis and  $m$  is its periodic length. Any  $n^{\text{th}}$  root  $g \in \text{Mod}^+(S)$  of  $h$  arises as a map taking  $\mathbf{T}_i$  to  $\mathbf{T}_{i+m/n}$ .  $\square$*

Given an acceptable certificate for  $h$ , we can compute a periodic part  $\mathbf{T}_i, \dots, \mathbf{T}_{i+m}$  in  $O(\text{poly}(\ell(h)))$  operations. From these we can compute all mapping classes induced by a map taking  $\mathbf{T}_i$  to  $\mathbf{T}_{i+m/n}$ , and so all  $n^{\text{th}}$  roots of  $h$ , in  $O(\text{poly}(\ell(h)))$  operations.

**Corollary 6.1.13.** *The function taking a word  $h \in X^*$  to:*

- $\{(g, n) : g \text{ is an } n^{\text{th}} \text{ root of } h\}$  if  $h$  is pseudo-Anosov, and
- the empty set otherwise

*is in **FNP**.*  $\square$

Hence there is an exponential time algorithm that, given a path  $p$  from  $\mathcal{T}$  to  $h(\mathcal{T})$ , computes the set of roots of  $h$  when  $h$  is pseudo-Anosov. This answers Question 3.27 of [45]. Of course from this we also get that deciding whether a pseudo-Anosov mapping class is *primitive*, that is, if it only has  $n^{\text{th}}$  roots when  $n = 1$ , is also a problem in  $\mathbf{NP} \cap \mathbf{co-NP}$ .

### 6.1.7 Special subgroups

One special class of subgroups of  $\text{Mod}^+(S)$  was first introduced by Hamidi-Tehrani [45]. To describe these using the terminology we have set up so far, for measured laminations  $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{ML}(S)$  we first define:

$$I(\mathcal{L}_1, \mathcal{L}_2) := \inf\{\varepsilon : \exists \gamma \in \mathcal{C}(S) \text{ such that } \mathcal{L}_i(\gamma) \leq \varepsilon \|\mathcal{T}(\mathcal{L}_i)\| \cdot \|\mathcal{T}(\gamma)\|\}.$$

This represents the measure assigned to  $\mathcal{L}_1$  by  $\mathcal{L}_2$  and vice versa and is independent of the chosen triangulation  $\mathcal{T}$ .

**Definition 6.1.14** ([45, Section 3.3]). A subgroup  $H \subseteq \text{Mod}^+(S)$  is *special* if there are pseudo-Anosov mapping classes  $g, h \in \text{Mod}^+(S)$  such that:



- $H \cong \langle g, h \rangle$ , and
- When  $\mathcal{L} \in \{\mathcal{L}^+(g), \mathcal{L}^+(g^{-1})\}$  and  $\mathcal{L}' \in \{\mathcal{L}^+(h), \mathcal{L}^+(h^{-1})\}$ ,

$$I(\mathcal{L}, \mathcal{L}') > \max(\lambda^-(g), \lambda^-(h)).$$

The original interest in these subgroups came from the fact that if  $H$  is special then a Ping-Pong argument shows that  $H \cong \mathbb{F}_2$  and every element of  $H$ , except for the identity element, is pseudo-Anosov [45, Proposition 3.21]. Furthermore, Hamidi-Tehrani showed that one can combine algorithms to:

1. find all of the roots of a pseudo-Anosov, and
2. find all of the periodic mapping classes fixing the stable lamination of a pseudo-Anosov

to give a solution to *the conjugacy problem for special subgroups* [45, Proposition 3.29]. That is, given special subgroups  $G, H \subseteq \text{Mod}^+(S)$ , decide whether there is a mapping class  $f \in \text{Mod}^+(S)$  such that  $fGf^{-1} = H$ . In our case, the exponential time algorithms to solve the root and commuting problems from Section 6.1.6 and Section 6.1.5 combine to give an exponential time algorithm to solve the conjugacy problem for special subgroups.

## 6.2 Implementation

The main algorithm has been implemented as part of the Python package **flipper** [11]. It includes an extensive array of tools for efficiently storing, manipulating and studying measured laminations via triangulations. **flipper** also includes a graphical user interface, as shown in Figure 6.2. Building such software appears as Problem 20 in Thurston’s list of open questions [86, Section 6]. Although **flipper** is still under active development it is being used by researchers around the world including Vincent Delecroix (LaBRI), Nathan Dunfield (UIUC), Vaibhav Gadre (Warwick), Ingrid Irmer (NUS), Jacob Rasmussen (Cambridge) and Dylan Thurston (Indiana).

By using the results of Section 6.1, **flipper** is also able to:

- determine the Nielsen–Thurston type of a mapping class,
- compute the dilatation of a mapping class,
- compute the stratum of a pseudo-Anosov,

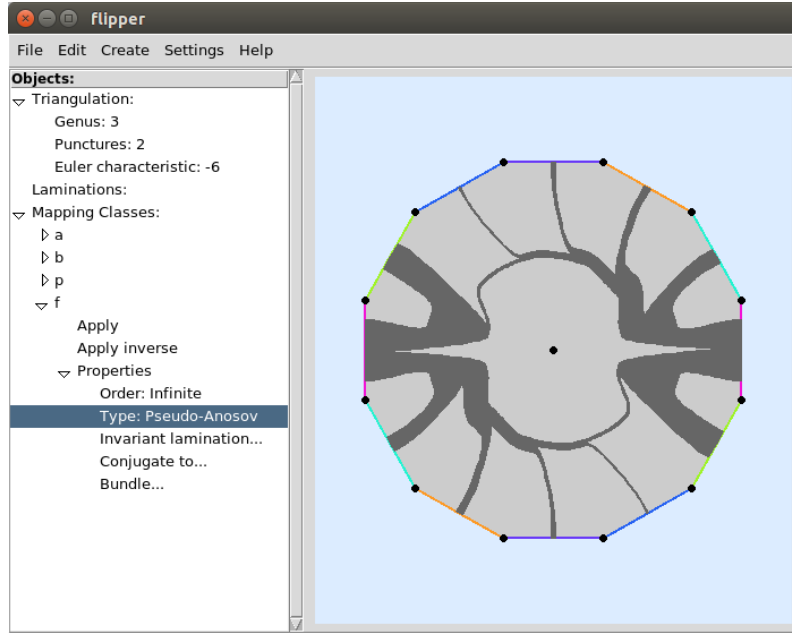


Figure 6.2: Checking a mapping class is pseudo-Anosov in **flipper**.

- decide whether the quadratic differential coming from a pseudo-Anosov mapping class is an Abelian differential,
- decide whether two mapping classes are conjugate whenever at least one of them is pseudo-Anosov,
- construct a triangulation of the mapping torus  $M_h$ , and
- construct Agol's veering triangulation [3, Section 4] of the mapping torus  $M_{h^\circ}$ , whenever  $h$  is a pseudo-Anosov mapping class and  $h^\circ$  is the mapping class on the surface with a marked point at each singularity of the stable foliation of  $h$ .

The calculations performed by **flipper** are exact and are not subject to floating point error. For example, for a  $k$ -bounded,  $m \times m$  integer matrix  $M$ , **flipper** uses Bareiss' algorithm [6] to compute  $\det(M)$  in

$$16m^5k^2$$

operations using only integer arithmetic. That is, any divisions that are performed are exact and without remainder.

Despite being entirely written in a high-level, dynamically typed, interpreted language, **flipper** is extremely high performance. It has been used to decide the Nielsen–Thurston type of mapping classes consisting of thousands of Dehn twists and

on surfaces of genus as high as 35. For additional performance, **flipper** includes an interface into the Sage Mathematics Software<sup>1</sup> [83].

For comparison, there are several implementations of the Bestvina–Handel algorithm. These include:

- **Trains** by Toby Hall [44],
- **BH** by Menasco and Ringland [64], and
- **XTrain** by Brinkmann [20] [21].

However, experimentally these implementations appear to be exponential in  $\ell(h)$  and so are significantly slower than **flipper** for any mapping classes of reasonable word length. See Appendix A for detailed timings of some examples.

As the constant  $K(S)$  of Theorem 4.2.10 is unknown, **flipper** implements the variant of the main algorithm described in Remark 4.3.2. Additionally, to avoid having to restart the entire process if the accuracy of an approximation is no longer sufficient to determine an algebraic number, **flipper** records the weights of  $T_i$  algebraically as integer linear combinations of the weights of  $T_0$ . This still only requires  $O(\text{poly}(\ell(h)))$  storage space but allows the weights of  $T_i$  to be immediately recalculated after  $T_0$  is approximated to a higher precision, without needing to recalculate all of  $T_1, \dots, T_{i-1}$ .

**flipper** includes several heuristics to help it find invariant laminations, each based on the idea of iteration. This allows it to avoid having to examine all of the exponentially many cells of the piecewise linear function corresponding to  $h$ . To do this **flipper** chooses a multicurve  $\gamma \in \mathcal{C}(S)$  such that  $\mathcal{T}(\gamma)$  is 1–bounded and looks for a projectively invariant lamination in the cells containing:

1.  $\mathcal{T}(h^i(\gamma))$ , and
2.  $\mathcal{T}(h^{5i}(\gamma)) - \mathcal{T}(h^{4i}(\gamma))$ ,

for various values of  $i$ .

When at least one of the canonical components of  $h$  is pseudo-Anosov the first heuristic is extremely effective as, after renormalising,  $h^i(\gamma) \rightarrow \mathcal{L}^+(h)$  exponentially. Otherwise, when all of the canonical components of  $h$  are periodic, there is a constant  $0 < k \leq \text{lcm}(1, \dots, 4\zeta)$ , depending only on  $S$ , such that  $h^k$  is a Dehn twist along a multicurve. By a result of Schaefer, Sedgwick and Štefankovič this second heuristic will exactly produce an invariant lamination when  $i = k$  [81, Lemma 7]. The modularity of **flipper** allows users to easily add additional functionality, including other heuristics for finding invariant laminations.

---

<sup>1</sup>Sage performs these calculations using PARI [85], FLINT [46], MPFI, the GNU MPFR Library [40] and the GNU Multiple Precision Arithmetic Library.

## 6.3 Examples

We give some examples of the types of calculations that can be performed using `flipper`. Each was performed on a single core of an Intel Celeron CPU 1007U running at 1.50 GHz from within Sage 6.5. For running times of even harder examples see Appendix A. For each of these examples we use SnapPy [31] and Regina [23] to verify that the mapping classes are the monodromies of the 3-manifolds that we claim them to be and Theorem 1.0.2 to ensure that we only ever manipulate hyperbolic manifolds with SnapPy.

### 6.3.1 On the once-marked torus

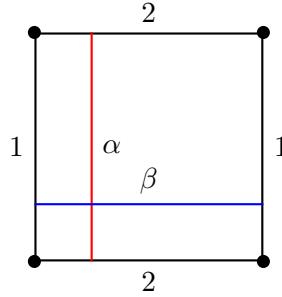


Figure 6.3: Curves on  $S_{1,1}$ .

When  $S = S_{1,1}$ , as  $\zeta = 3$ , the action of  $h$  is described by  $3 \times 3$  matrices. Hence  $\lambda^+(h)$  is an eigenvalue of a  $3 \times 3$  integer matrix, the other eigenvalues of which are  $\lambda^-(h)$  and 1. Therefore  $\mathbb{Q}(\lambda^+(h))$ , which contains  $\mathcal{L}^+(h)(\alpha)$  for any arc  $\alpha$ , is a quadratic extension and so calculations are significantly faster. Now any train track on  $S$  has at most one splittable branch we may strengthen Theorem 4.2.12 to:

**Proposition 6.3.1.** *The preperiodic length of  $T$ , a train track representing  $\mathcal{L}^+(h)$ , is at most  $2Kd(T, \widehat{h}(T))$ .  $\square$*

By the same argument, this bound also holds for  $S_{0,4}$ .

**Conjecture 6.3.2.** *For any marked surface  $S$ , the preperiodic length of  $T$ , a train track representing  $\mathcal{L}^+(h)$ , is at most  $O(d(T, \widehat{h}(T)))$ .*

Consider the curves on  $S$  shown in Figure 6.3. The following script uses `flipper` to repeat the calculations of Chapter 1. It verifies that  $h := T_\alpha \circ T_\beta^{-1}$ , the mapping class corresponding to the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

is pseudo-Anosov and the monodromy of the figure 8 knot complement. It goes on to check that the dilatation of  $h$  is  $(1 + \sqrt{5})/2 \approx 2.618034$  and that  $h$  is primitive. It finishes by using a result of Burde, Zieschang and Heusener that a fibred knot is amphichiral if and only if the monodromy is conjugate to its inverse [22, Proposition 5.12] to verify that the figure 8 knot is amphichiral.

```

1 | import flipper
2 | from snappy import Manifold
3 |
4 | S = flipper.load('S_1_1')
5 | h = S.mapping_class('aB')
6 |
7 | # Check that M_h is hyperbolic...
8 | assert(h.is_pseudo_anosov())
9 | print('h is pseudo-Anosov.')
10 |
11 | # ...and actually the figure 8 knot.
12 | M_h = Manifold(h.bundle().snappy_string())
13 | assert(M_h.is_isometric_to(Manifold('4_1')))
14 | print('M_h is the figure 8 knot.')
15 |
16 | print('h has dilatation %f' % h.dilatation())
17 | print('this is an algebraic number with minimal polynomial %s' %
    |     ↪ h.dilatation().minimal_polynomial())
18 | assert(h.is_primitive())
19 | print('h is primitive.')
20 | assert(h.is_conjugate_to(h.inverse()))
21 | print('h is amphichiral.')

```

### 6.3.2 On the twice-marked torus

Parker and Series used a similar coordinate system coming from  $\pi_1$ -train tracks to study the mapping class group of  $S = S_{1,2}$  [73]. They explicitly constructed the piecewise linear maps for a generating set of  $\text{PMod}^+(S)$ . Menzel and Parker went on to use this coordinate system to give an algorithm to decide whether a mapping class is pseudo-Anosov [65].

Consider the curves on  $S$  shown in Figure 6.4. The mapping class  $h := T_\alpha \circ T_\beta \circ T_\gamma^{-1}$  is the monodromy of the Whitehead link, shown in Figure 6.5. As an example, Menzel and Parker computed several properties of  $h$  [65, Page 14]. In particular they showed that  $h$  is pseudo-Anosov and that

$$\lambda^+(h) = \frac{1 + \sqrt{3} + \sqrt[4]{12}}{2} \approx 2.29663026289$$

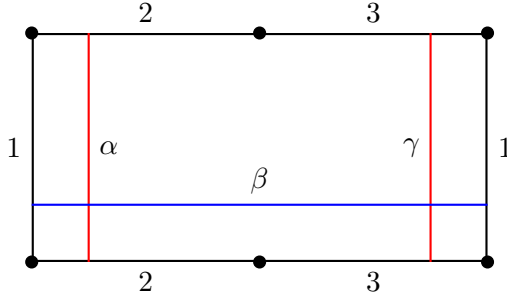


Figure 6.4: Curves on  $S_{1,2}$ .

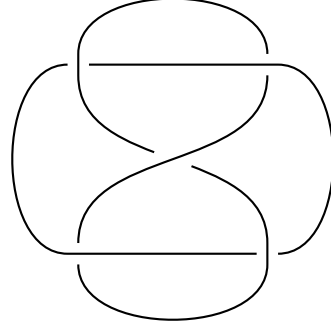


Figure 6.5: The Whitehead link.

We can repeat their calculations using `flipper` and the following Python script:

```

1 | import flipper
2 | from snappy import Manifold
3 |
4 | S = flipper.load('S_1_2')
5 | h = S.mapping_class('abC')
6 |
7 | # Check that M_h is hyperbolic...
8 | assert(h.is_pseudo_anosov())
9 | print('h is pseudo-Anosov.')
10 |
11 | # ...and actually the Whitehead link.
12 | M_h = Manifold(h.bundle().snappy_string())
13 | assert(M_h.is_isometric_to(Manifold('5_2_1')))
14 | print('M_h is the Whitehead link.')
15 |
16 | print('h has dilatation %f' % h.dilatation())
17 | print('this is an algebraic number with minimal polynomial %s' %
    |     ↪ h.dilatation().minimal_polynomial())

```

### 6.3.3 On a higher genus surface

Consider the chain of curves on  $S_{2,1}$  shown in Figure 6.6 and let  $h := T_\alpha \circ T_\beta \circ T_\gamma \circ T_\delta^{-1} \circ T_\epsilon^{-1}$ . The following Python script uses `flipper` to verify that the mapping torus  $M_h$  is the  $8_{20}$  knot complement and moreover that the singularity of  $\mathcal{L}^+(h)$  at the marked point has order one. Therefore the mapping class  $h^\bullet \in \text{Mod}^+(S_{2,0})$ , obtained by removing the marked point of  $S$ , is not automatically pseudo-Anosov. The script goes on to use Regina [23] to check  $h^\bullet$  is reducible by checking that the mapping torus  $M_{h^\bullet}$  contains an incompressible torus. This can also be seen in the

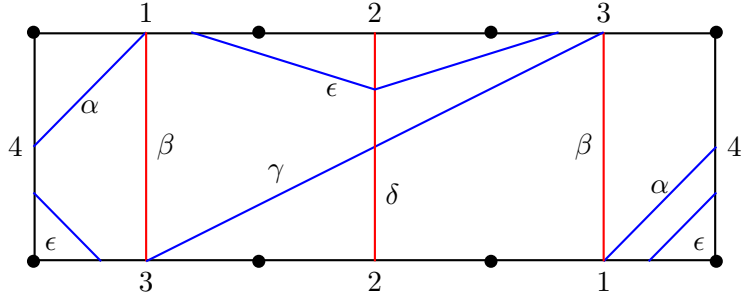


Figure 6.6: Curves on  $S_{2,1}$ .

fundamental group of a six-fold cover of  $M_{h\bullet}$ :

$$\pi_1(M_{(h\bullet)^6}) = \left\langle a, b, c, d, e \left| \begin{array}{lll} [a, e], & [c, e], & ac^{-1}ac^{-1}e^{-3}ba^2eb^{-1} \\ a^2d^{-1}c^2ede, & ba^{-1}d^{-1}c^3e^2db^{-1}ca^{-1}e^2 & \end{array} \right. \right\rangle.$$

That is, the fibre slope of the  $8_{20}$  knot complement is an *exceptional* slope.

```

1 import flipper
2 from snappy import Manifold
3
4 S = flipper.load('S_2_1')
5 # Note that the curves have different names in the standard flipper example.
6 h = S.mapping_class('abCDf')
7
8 # Check that M_h is hyperbolic.
9 assert(h.is_pseudo_anosov())
10 print('h is pseudo-Anosov.')
11
12 B = h.bundle()
13 M_h = Manifold(B.snappy_string())
14 # and actually 8_20.
15 assert(M_h.is_isometric_to(Manifold('8_20')))
16 print('M_h is the 8_20 knot.')
17
18 # Get the order of the marked point.
19 stratum = h.stratum()
20 marked_point_order = [stratum[sing] for sing in stratum if not sing.filled][0]
21 print('Marked point has order %d' % marked_point_order)
22
23 # Check that there is an incompressible torus
24 # after filling the fibre slope.
25 # This requires Regina-Python.
26 M_h.dehn_fill(B.fibre_slopes())
27 T = NTriangulation(M_h.filled_triangulation()._to_string())
28 surfaces = NNormalSurfaceList.enumerate(T, NormalCoords.NS_QUAD)
29 for i in range(surfaces.getNumberOfSurfaces()):
30     surface = surfaces.getSurface(i)
31     if surface.isCompact() and surface.isOrientable() and surface.isTwoSided():
32         if surface.getEulerCharacteristic() == 0:
33             if not surface.cutAlong().hasCompressingDisc():
34                 print('After filling, M_h contains an incompressible torus.')
35                 break
36 else:
37     assert(False)

```

### 6.3.4 Penner like examples

Consider the curves on  $S_{4,2}$  shown in Figure 6.7 and let  $\rho \in \text{Mod}^+(S)$  be the order 16 rotation about the centre, taking  $\alpha$  to  $\beta$ . The Dehn twists about  $\alpha$  and  $\beta$  generate the mapping class group of the subsurface that they fill. Penner used mapping classes



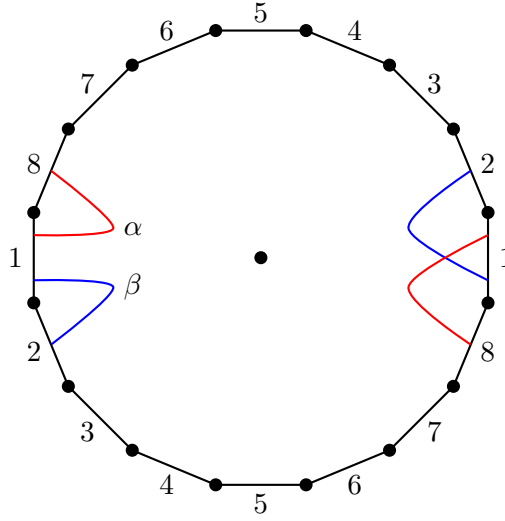


Figure 6.7: Curves on  $S_{4,2}$ .

of the form  $\rho \circ h$ , where  $h \in \langle T_\alpha, T_\beta \rangle$  to give upper bounds on the least dilatation pseudo-Anosovs [74].

Again we can use **flipper** to determine the Nielsen–Thurston type of various mapping classes from various families, for example see Table 6.1 and Table 6.2. The stable lamination of one of these examples is shown in Figure 6.8.

|     |    | $j$ |    |    |    |    |    |    |    |    |
|-----|----|-----|----|----|----|----|----|----|----|----|
|     |    | -4  | -3 | -2 | -1 | 0  | 1  | 2  | 3  | 4  |
| $i$ | -4 | pA  | pA | pA | pA | pA | pA | pA | R  | P  |
|     | -3 | pA  | pA | pA | pA | pA | pA | R  | P  | R  |
|     | -2 | pA  | pA | pA | pA | pA | R  | P  | R  | pA |
|     | -1 | pA  | pA | pA | pA | R  | P  | R  | pA | pA |
|     | 0  | pA  | pA | pA | R  | P  | R  | pA | pA | pA |
|     | 1  | pA  | pA | R  | P  | R  | pA | pA | pA | pA |
|     | 2  | pA  | R  | P  | R  | pA | pA | pA | pA | pA |
|     | 3  | R   | P  | R  | pA | pA | pA | pA | pA | pA |
|     | 4  | P   | R  | pA | pA | pA | pA | pA | pA | pA |

Table 6.1: The Nielsen–Thurston type of  $\rho \circ T_\beta^j \circ T_\alpha^i$ .  
In every pseudo-Anosov case  $\mathcal{L}^+$  is orientable.

This type of experimental evidence leads to conjectures like:

**Conjecture 6.3.3.** *Whenever  $|i+j| > 1$ , the mapping class  $\rho \circ T_\beta^j \circ T_\alpha^i$  is pseudo-Anosov and its stable lamination is orientable.*

|     |    | -4  | -3  | -2  | -1  | $j$<br>0 | 1   | 2   | 3   | 4   |
|-----|----|-----|-----|-----|-----|----------|-----|-----|-----|-----|
|     | -4 | pA  | pA  | pA  | pA  | pA*      | pA* | pA* | pA* | pA* |
|     | -3 | pA  | pA  | pA  | pA  | pA*      | pA* | pA* | pA* | pA* |
|     | -2 | pA  | pA  | pA  | pA  | R        | pA* | pA* | pA* | pA* |
|     | -1 | pA  | pA  | pA  | R   | P        | pA* | pA* | pA* | pA* |
| $i$ | 0  | pA* | pA* | R   | P   | R        | pA* | pA* | pA* | pA* |
|     | 1  | pA* | pA* | pA* | pA* | pA*      | R   | pA  | pA  | pA  |
|     | 2  | pA* | pA* | pA* | pA* | pA*      | pA  | pA  | pA  | pA  |
|     | 3  | pA* | pA* | pA* | pA* | pA*      | pA  | pA  | pA  | pA  |
|     | 4  | pA* | pA* | pA* | pA* | pA*      | pA  | pA  | pA  | pA  |

Table 6.2: The Nielsen–Thurston type of  $\rho \circ T_\beta^j \circ T_\alpha^i \circ T_\beta$ .  
Only in the pseudo-Anosov cases marked with a \* is  $\mathcal{L}^+$  orientable.

## 6.4 Further extensions

We finish with some further remarks about generalisations of the results of the previous chapters.

### 6.4.1 The extended mapping class group

All of the techniques that we have described can also be applied to the *extended mapping class group*  $\text{Mod}^\pm(S)$ , which includes orientation-reversing mapping classes [38, Page 219]. We note however that Dehn twists and half-twists are insufficient to generate  $\text{Mod}^\pm(S)$  and an additional orientation reversing mapping class is needed.

### 6.4.2 Other surfaces

There are several generalisations of the surfaces we have been working with. Firstly, when  $S$  is an unmarked surface we can still produce the results of Chapters 3 and 4 by switching to a different coordinate system. For example, the Dehn–Thurston coordinates on  $\mathcal{C}(S)$  and  $\mathcal{ML}(S)$  coming from a pants decomposition [39, Section 6.4]. As  $\text{Mod}^+(S)$  also acts piecewise linearly on  $\mathcal{ML}(S)$  with respect to these coordinates [76] the results and algorithms of Chapters 3 and 4 can be directly reconstructed.

However, in its current form, the periodic case of the conjugacy problem requires at least one puncture in order to determine the quotient orbifold. This is an obstruction to constructing the total conjugacy invariant in this case.

**Question 6.4.1.** Suppose that  $S$  is a closed surface. Is deciding whether two periodic words  $g, h \in X^*$  are conjugate a problem in **co-NP**?

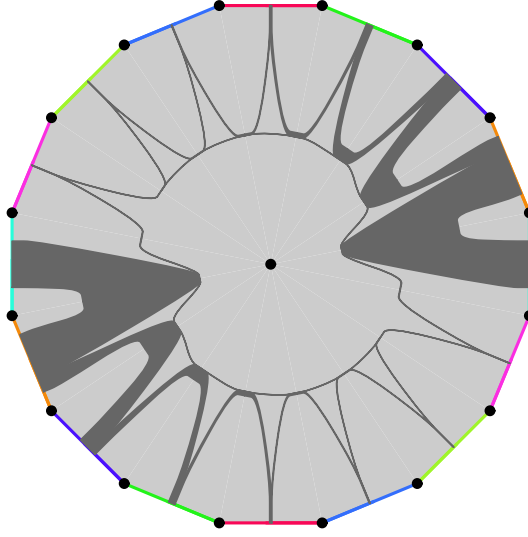


Figure 6.8: The stable lamination for the pseudo-Anosov mapping class  $\rho \circ T_\alpha^2 \circ T_\beta^{-3}$ , as found by `flipper`.

A second alternative is to allow  $S$  to be a surface with boundary. For these we require that mapping classes fix the boundary components pointwise. In this case, by adding a single marked point to each boundary component we can again triangulate  $S$  and reproduce the algorithms of Chapters 3 and 4. In this case a slight modification to the matrices  $F_i$  of Lemma 3.3.1 is needed to disallow peripheral curves.

A third possibility is to allow the surface to be non-orientable. We highlight the fact that, as well as additional generators, extra care is needed. For example, if  $\gamma \in \mathcal{C}(S)$  is a curve then  $\iota(\gamma, \mathcal{T}) = 2$  is no longer sufficient to deduce that  $\gamma$  meets  $\mathcal{T}$  as shown in Figure 2.3.

### 6.4.3 Independence of surface

It is also interesting to consider how the difficulty of these problems changes when the surface is also allowed to vary. As before, we specify the surface via a triangulation and so the complexity of the specification is  $O(\zeta)$ .

**Problem 6.4.2** (The generalised reducibility problem). Given a triangulation  $\mathcal{T}$  (of an arbitrary marked surface) and a path  $p$  from  $\mathcal{T}$  to  $h(\mathcal{T})$ , decide whether  $h$  is reducible.

For this problem, the bounds constructed in Chapter 3 still hold in this more general setting.

**Proposition 6.4.3.** *If  $h$  is reducible then there is a  $h$ -invariant multicurve  $\gamma$  such that  $\mathcal{T}(\gamma)$  is  $O(\ell(h) + \zeta)$ -bounded.*  $\square$

Again such a multicurve acts as a certificate that  $h$  is reducible and is sufficiently small that we obtain that:

**Corollary 6.4.4.** *The generalised reducibility problem is in **NP**.*  $\square$

**Problem 6.4.5** (The generalised pseudo-Anosov problem). Given a path  $p$  from  $\mathcal{T}$  to  $h(\mathcal{T})$  where  $\mathcal{T}$  is a triangulation of an arbitrary marked surface, decide whether  $h$  is pseudo-Anosov.

We note that the quantities  $h_0$ ,  $h_1$ ,  $t$ ,  $d_1$  and  $p_1$  of the main algorithm are all also polynomial functions of  $\zeta$  and  $K(S)$ , where  $K(S)$  is the constant of Theorem 4.2.10. Therefore, in this more general setting, the main algorithm will run in  $O(\text{poly}(\ell(p) + \zeta + K(S)))$  operations.

**Corollary 6.4.6.** *If  $K(S) \in O(\text{poly}(\zeta(S)))$  then the generalised pseudo-Anosov problem is in **NP**.*  $\square$

**Problem 6.4.7** (The generalised conjugacy problem). Given paths  $p$  from  $\mathcal{T}$  to  $g(\mathcal{T})$  and  $q$  from  $\mathcal{T}'$  to  $h(\mathcal{T}')$  where  $\mathcal{T}$  and  $\mathcal{T}'$  are triangulations of an arbitrary marked surface, decide whether  $g$  and  $h$  are conjugate mapping classes.

This generalisation appears to be significantly harder.

**Proposition 6.4.8.** *The generalised conjugacy problem is at least as hard as the graph isomorphism problem, that is, the problem of deciding whether two graphs are isomorphic [82, Page 355].*

*Proof.* Consider two finite graphs  $G$  and  $H$ . We will assume that each has  $v$  vertices (each of degree at least three) and  $e$  edges. Choose embeddings of  $G$  and  $H$  into  $\mathbb{R}^3$ . Let  $S := \partial N(G)$  and  $S' := \partial N(H)$ , with a marked point added to  $S$  (respectively  $S'$ ) corresponding to each vertex of  $G$  (reps.  $H$ ). Let  $\gamma$  (respectively  $\gamma'$ ) be the multicurve on  $S$  obtained by lifting the midpoint of each edge of  $G$  (resp.  $H$ ) to a curve. Let  $g := T_\gamma \in \text{Mod}^+(S)$  and  $h := T_{\gamma'} \in \text{Mod}^+(S')$ , then  $g$  is conjugate to  $h$  if and only if  $G$  is isomorphic to  $H$ . Moreover

$$\zeta = \zeta(S) = \zeta(S') = 18e + 15v - 6$$

and we may choose triangulations  $\mathcal{T}$  and  $\mathcal{T}'$  of  $S$  and  $S'$  respectively such that  $\iota(\gamma, \mathcal{T}), \iota(\gamma', \mathcal{T}') \in O(\zeta^2)$ . Hence there are paths  $p$  and  $p'$  representing  $g$  and  $h$  such that  $\ell(p), \ell(p') \in O(\zeta^3)$  and so the generalised conjugacy problem is at least as hard as the graph isomorphism problem.  $\square$

Although the graph isomorphism problem is known to be in **NP**, it is not known to be in **P**, **NP**-complete or even **co-NP** [82, Page 355]. It remains a problem of considerable interest.

# Appendix A

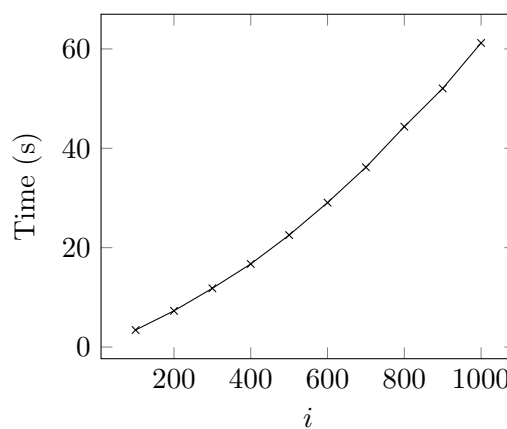
## flipper timings

### A.1 On the once-marked torus

The following `flipper` script decides the Nielsen–Thurston type of a sequence of mapping classes on  $S_{1,1}$ . The running times of these calculations are also shown below.

```
1 import flipper
2 from time import time
3
4 S = flipper.load('S_1_1')
5
6 for i in [100, 200, 300, 400, 500, 600, 700, 800, 900, 1000]:
7     start_time = time()
8     h = S.mapping_class('a' * i + 'B' * i)
9     assert(h.is_pseudo_anosov())
10    print(i, time() - start_time)
```

| $i$  | Time (s) |
|------|----------|
| 100  | 3.43     |
| 200  | 7.29     |
| 300  | 11.82    |
| 400  | 16.72    |
| 500  | 22.53    |
| 600  | 29.07    |
| 700  | 36.18    |
| 800  | 44.35    |
| 900  | 52.04    |
| 1000 | 61.20    |



## A.2 On the twice-marked torus

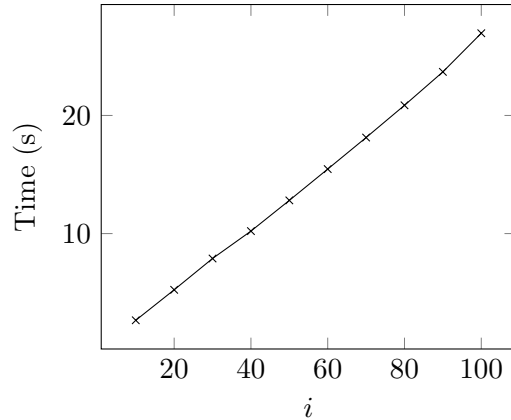
The following `flipper` script decides the Nielsen–Thurston type of a sequence of mapping classes on  $S_{1,2}$ . The running times of these calculations are also shown below.

```

1 | import flipper
2 | from time import time
3 |
4 | S = flipper.load('S_1_2')
5 |
6 | for i in [10, 20, 30, 40, 50, 60, 70, 80, 90, 100]:
7 |     start_time = time()
8 |     h = S.mapping_class('a' * i + 'B' * i + 'C' * i)
9 |     assert(h.is_pseudo_anosov())
10 |    print(i, time() - start_time)

```

| $i$ | Time (s) |
|-----|----------|
| 10  | 2.66     |
| 20  | 5.24     |
| 30  | 7.89     |
| 40  | 10.21    |
| 50  | 12.81    |
| 60  | 15.46    |
| 70  | 18.13    |
| 80  | 20.86    |
| 90  | 23.69    |
| 100 | 26.96    |



## A.3 A comparison with Bestvina–Handel implementations

We give some comparisons between `flipper` and the available implementations of the Bestvina–Handel algorithm [44] [64] [21]. To help ensure a fair comparison, we compare functionality that is common to every implementation. Thus we look at the time taken to decide the Nielsen–Thurston type of various braids  $\beta \in B_n$ . The results of which are shown in Figure A.1. Note that these plots are all done on a semi-log scale due to the drastic difference in running times.

These examples emphasise the exponential nature of the Bestvina–Handel algorithm. Hence, for braids consisting of more than  $\sim 30$  twists, `flipper` appears to be significantly faster than any of these implementations.

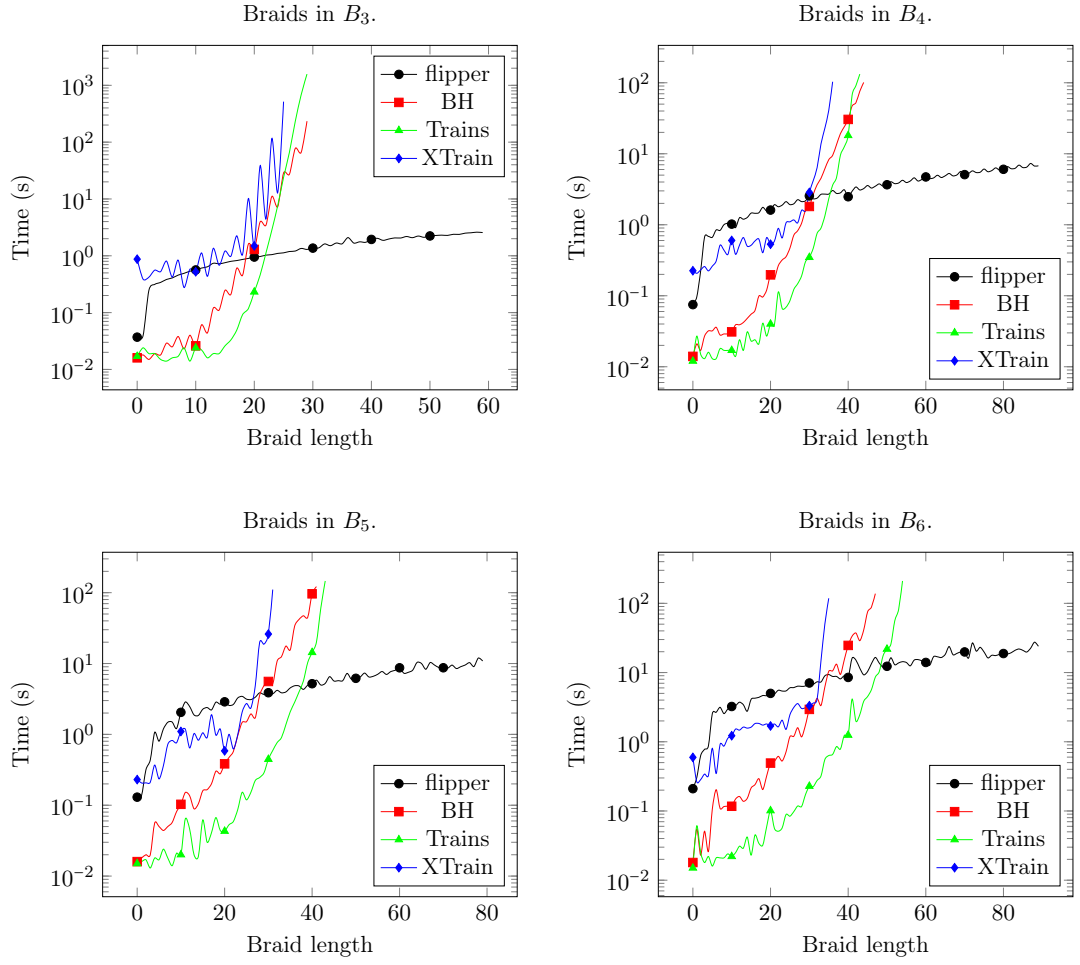


Figure A.1: A comparison of running times of **flipper** and implementations of the Bestvina–Handel algorithm on random braids.



# Appendix B

## flipper source code

In this appendix we highlight two key sections of code from the `flipper` kernel.

### B.1 Invariant laminations

The method of a mapping class for finding a projectively invariant lamination. Taken from `flipper.kernel.encoding`.

```
1 def invariant_lamination(self):
2     ''' Return a rescaling constant and projectively invariant lamination.
3
4     Assumes that the mapping class is pseudo-Anosov.
5
6     To find this we start with a curve on the surface and repeatedly apply
7     the map until it appear to be projectively similar to a previous iteration.
8     Finally it uses:
9         flipper.kernel.symboliccomputation.perron_frobenius_eigen()
10    to find the nearby projective fixed point. If it cannot find one then it
11    raises a ComputationError.
12
13    Note: In most pseudo-Anosov cases < 15 iterations are needed, if it fails
14    to converge after many iterations and a ComputationError is thrown then it
15    is extremely likely that the mapping class was not pseudo-Anosov.
16
17    This encoding must be a mapping class. '''
18
19    # Suppose that f_1, ..., f_m, g_1, ..., g_n, t_1, ..., t_k, p is the Thurston
20    #   ↳ decomposition of self.
21    # That is: f_i are pA on subsurfaces, g_i are periodic on subsurfaces, t_i are
22    #   ↳ Dehn twist along the curve of
23    # the canonical curve system and p is a permutation of the subsurfaces.
24    # Additionally, let S_i be the subsurface corresponding to f_i, P_i correspond to
25    #   ↳ g_i and A_i correspond to t_i.
26    # Finally, let x_0 be a curve on the surface and define x_i := self(x_{i-1}).
27    #
```

```

25 # The algorithm covers 3 cases: (Note we reorder the subsurfaces for ease of
    ↳ notation.)
26 # 1)  $x_0$  meets at  $S_1, \dots, S_m$ ,
27 # 2)  $x_0$  meets no  $S_i$  but meets  $A_1, \dots, A_k$ , and
28 # 3)  $x_0$  meets no  $S_i$  or  $A_i$ , that is,  $x_0$  is contained in a  $P_1$ .
29 #
30 # In the first case,  $x_i$  will converge exponentially to the stable laminations of
    ↳  $f_1, \dots, f_m$ .
31 # Here the convergence is so fast we need only a few iterations.
32 #
33 # In the second case  $x_i$  will converge linearly to  $c$ , the cores of  $A_1, \dots$ ,
    ↳  $A_k$ . To speed this up
34 # we note that  $x_i = i*c + O(1)$ , so rescale  $x_i$  by  $1/i$ , round and check if this
    ↳ is  $c$ .
35 #
36 # Finally, the third case happens if and only if  $x_i$  is periodic. In this case
    ↳ self must be
37 # periodic or reducible. We test for periodicity at the beginning hence if we
    ↳ ever find a curve
38 # fixed by a power of self then we must reducible.
39
40 assert(self.is_mapping_class())
41
42 # We start with a fast test for periodicity.
43 # This isn't needed but it means that if we ever discover that
44 # self is not pA then it must be reducible.
45 if self.is_periodic():
46     raise flipper.AssumptionError('Mapping class is periodic.')
47
48 triangulation = self.source_triangulation
49 max_order = triangulation.max_order
50 curves = [triangulation.key_curves()[0]]
51
52 # A little helper function to determine how much two vectors differ by.
53 def projective_difference(A, B, error_reciprocal):
54     ''' Return if the projective difference between A and B is less than 1 /
        ↳ error_reciprocal. '''
55
56     A_sum, B_sum = sum(A), sum(B)
57     return max(abs((p * B_sum) - (q * A_sum)) for p, q in zip(A, B)) *
        ↳ error_reciprocal < A_sum * B_sum
58
59 # We will remember the cells we've tested to avoid recalculating their
    ↳ eigenvectors again.
60 for i in range(100):
61     new_curve = self(curves[-1])
62     # print(new_curve)
63
64     # Check if we have seen this curve before.
65     if new_curve in curves: # self**(i-j)(curve) == curve, so self is reducible.
66         raise flipper.AssumptionError('Mapping class is reducible.')
67
68     curves.append(new_curve)
69     for j in range(1, min(max_order, len(curves))):
70         old_curve = curves[-j-1]

```

```

71     if projective_difference(new_curve, old_curve, 100):
72         average_curve = sum(curves[-j:])
73         action_matrix, condition_matrix = (self**j).applied_geometric(average_curve)
74     try:
75         eigenvalue, eigenvector =
76             ↳ flipper.kernel.symboliccomputation.directed_eigenvector(
77                 action_matrix, condition_matrix, average_curve)
78     except flipper.ComputationError:
79         pass # Could not find an eigenvector in the cone.
80     except flipper.AssumptionError:
81         raise flipper.AssumptionError('Mapping class is reducible.')
82     else:
83         # Test if the vector we found lies in the cone given by the condition
84         ↳ matrix.
85         # We could also use:
86         ↳ invariant_lamination.projectively_equal(self(invariant_lamination))
87         # but this is much faster.
88         if flipper.kernel.matrix.nonnegative(eigenvector) and
89             ↳ condition_matrix.nonnegative_image(eigenvector):
90             # If it does then we have a projectively invariant lamination.
91             invariant_lamination = triangulation.lamination(eigenvector)
92             if not invariant_lamination.is_empty(): # But it might have been
93                 ↳ entirely peripheral.
94             if j == 1:
95                 # We could raise an AssumptionError as this actually shows that self
96                 ↳ is reducible.
97                 return eigenvalue, invariant_lamination
98             else:
99                 if not
100                     ↳ invariant_lamination.projectively_equal(self(invariant_lamination)):
101                     raise flipper.AssumptionError('Mapping class is reducible.')
102                 else:
103                     # We possibly could reconstruct something here but all the numbers
104                     ↳ are
105                     # in the wrong number field. It's easier to just keep going.
106                     pass
107         break
108
109     # See if we are close to an invariant curve.
110     # Build some different vectors which are good candidates for reducing curves.
111     vectors = [[x - y for x, y in zip(new_curve, old_curve)] for old_curve in
112                 ↳ curves[max(len(curves) - max_order, 0):]]
113
114     for vector in vectors:
115         new_small_curve = small_curve = triangulation.lamination(vector,
116             ↳ algebraic=[0] * self.zeta)
117         if not small_curve.is_empty():
118             for j in range(1, max_order+1):
119                 new_small_curve = self(new_small_curve)
120                 if new_small_curve == small_curve:
121                     if j == 1:
122                         # We could raise an AssumptionError in this case too as this also
123                         ↳ shows that self is reducible.
124                         return 1, small_curve
125                     else:

```

```

115         raise flipper.AssumptionError('Mapping class is reducible.')
116
117     raise flipper.ComputationError('Could not estimate invariant lamination.')

```

## B.2 Splitting sequence

The method of a measured lamination for computing its maximal splitting sequence. Taken from `flipper.kernel.lamination`.

```

1 def splitting_sequences(self, dilatation=None):
2     ''' Return a list of splitting sequence associated to this lamination.
3
4     This is the encoding obtained by flipping edges to repeatedly split
5     the branches of the corresponding train track with maximal weight
6     until you reach a projectively periodic sequence (with the required
7     dilatation if given).
8
9     Assumes that this lamination is projectively invariant under some mapping class.
10    Assumes (and checks) that this lamination is filling.
11
12    Each entry of self.geometric must be an Integer or a NumberFieldElement (over
13    the same NumberField). '''
14
15    # In this method we use Lamination.projective_hash to store the laminations
16    # we encounter efficiently and so avoid a quadratic algorithm. This currently
17    # only ever uses the default precision HASH_DENOMINATOR. At some point this
18    # should change dynamically depending on the algebraic numbers involved in
19    # this lamination.
20
21    assert(all(isinstance(entry, flipper.IntegerType) or isinstance(entry,
22        ↪ flipper.kernel.NumberFieldElement) for entry in self))
23    assert(len(set([entry.number_field for entry in self if isinstance(entry,
24        ↪ flipper.kernel.NumberFieldElement)])) <= 1)
25
26    # Check if the lamination is obviously non-filling.
27    if any(v == 0 for v in self):
28        raise flipper.AssumptionError('Lamination is not filling.')
29
30    if all(isinstance(v, flipper.IntegerType) for v in self):
31        raise flipper.AssumptionError('Lamination is not filling.')
32
33    # Puncture all the triangles where the lamination is a tripod.
34    E = self.puncture_tripods() if len(self.tripod_regions()) > 0 else
35        ↪ self.triangulation.id_encoding()
36    lamination = E(self)
37
38    encodings = [E]
39    laminations = [self, lamination]
40    num_isometries = [len(lamination.self_isometries())]
41    seen = {lamination.projective_hash(): [1]}
42    # We start indexing at 1 to help keep the indices aligned.

```

```

40 # We don't want to include self as the first lamination just in case
41 # it is already on the axis and the puncture_tripods does nothing,
42 # misaligning the indices.
43 while True:
44     # Find the index of the largest entry.
45     edge_index = max(range(lamination.zeta), key=lambda i: lamination[i])
46     E = lamination.triangulation.encode_flip(edge_index)
47     encodings.append(E)
48     lamination = E(lamination)
49     laminations.append(lamination)
50     num_isometries.append(len(lamination.self_isometries()))
51
52     # Check if we have created any edges of weight 0.
53     # It is enough to just check edge_index.
54     if lamination[edge_index] == 0:
55         try:
56             # If this fails it's because the lamination isn't filling.
57             lamination, E2 = lamination.collapse_trivial_weight(edge_index)
58             encodings.append(E2)
59             laminations.append(lamination)
60             num_isometries.append(len(lamination.self_isometries()))
61         except flipper.AssumptionError:
62             raise flipper.AssumptionError('Lamination is not filling.')
63
64     # In the next block we have a lot of tests to do. We'll do these in
65     # order of difficulty of computation. For example, computing
66     # projective_isometries is slow; so we'll leave that to last to give
67     # us the best chance that a faster test failing will allow us to
68     # skip it.
69
70     # Check if it (projectively) matches a lamination we've already seen.
71     target = lamination.projective_hash()
72     if target in seen:
73         for index in seen[target]:
74             # We need to stop at a point with maximal symmetry.
75             if num_isometries[-1] == max(num_isometries[index:]):
76                 old_lamination = laminations[index]
77                 if dilatation is None or old_lamination.weight() >= dilatation *
78                     ↪ lamination.weight():
79                     isometries = lamination.all_projective_isometries(old_lamination)
80                     if len(isometries) > 0:
81                         assert(old_lamination.weight() == dilatation * lamination.weight())
82                         return [flipper.kernel.SplittingSequence(encodings + [isom.encode()],
83                             ↪ index, dilatation, laminations[index]) for isom in isometries]
84             else:
85                 # dilatation is not None and:
86                 # old_lamination.weight() < dilatation * lamination.weight():
87                 # Note that the weight of laminations is strictly decreasing and the
88                 # indices of seen[target] are increasing. Thus if we are in this case
89                 # then the same inequality holds for every later index in seen[target].
90                 # Hence we may break out.
91                 break
92         seen[target].append(len(laminations)-1)
93     else:
94         seen[target] = [len(laminations)-1]

```

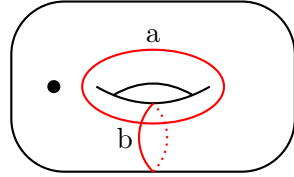
# Appendix C

## Censuses

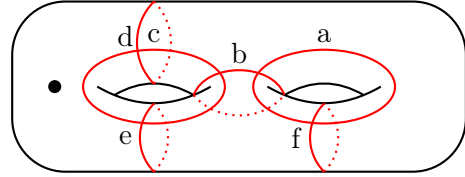
For each surface in Figure C.1, we use the shown collection of curves as a generating set of  $\text{Mod}^+(S)$ . Many fibre bundles over these surfaces appear in censuses of 3-manifolds [50] [28] [24]. We list examples of monodromies of such manifolds, along with several of their key properties. Additional data can be found in [10].

To save space, we use the following conventions:

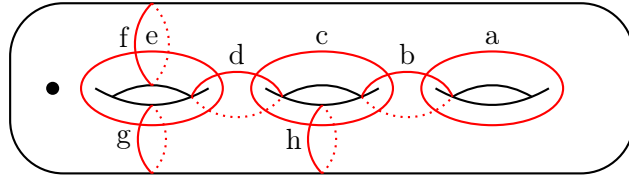
- We abbreviate  $T_x$  to  $\mathbf{x}$  and  $T_x^{-1}$  to  $\mathbf{X}$  and denote a composition of Dehn twists by the corresponding word. For example,  $T_a \circ T_b^{-1} \circ T_c$  is denoted  $\mathbf{aBc}$ .
- We truncate polynomials when they are symmetric.
- We group singularity orders together, denoting their multiplicity with a power.
- We mark singularity orders with a  $*$  if they occur at a marked point.



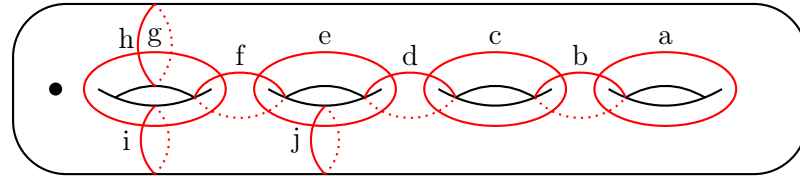
(a) Curve collection for  $S_{1,1}$ .



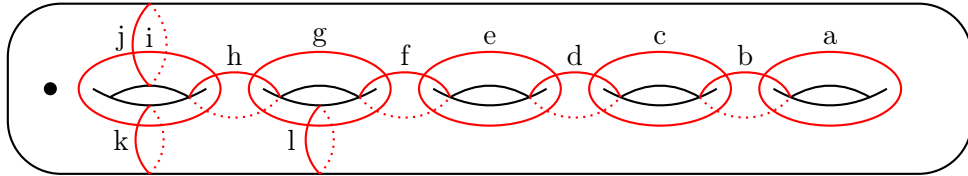
(b) Curve collection for  $S_{2,1}$ .



(c) Curve collection for  $S_{3,1}$ .



(d) Curve collection for  $S_{4,1}$ .



(e) Curve collection for  $S_{5,1}$ .

Figure C.1: Generators of  $\text{Mod}^+(S)$ .

## C.1 Monodromies of fibred knots

The manifolds in the following tables appear in the Hoste–Thistlethwaite–Weeks knot tables [50] [28].

Table C.1: Fibred knot complements with fibre  $S_{1,1}$ .

| Name  | Monodromy | $\lambda(h)$ | $\mu_{\lambda(h)}(x)$ | Stratum   | Stable lamination |
|-------|-----------|--------------|-----------------------|-----------|-------------------|
| $4_1$ | aB        | 2.618033...  | $x^2 - 3x + \dots$    | $\{2^*\}$ | Orientable        |

Table C.2: Fibred knot complements with fibre  $S_{2,1}$ .

| Name       | Monodromy | $\lambda(h)$ | $\mu_{\lambda(h)}(x)$               | Stratum           | Stable lamination |
|------------|-----------|--------------|-------------------------------------|-------------------|-------------------|
| $6_2$      | abcD      | 2.153721...  | $x^4 - 3x^3 + 3x^2 + \dots$         | $\{2^*, 4^2\}$    | Orientable        |
| $6_3$      | abCD      | 2.015357...  | $x^6 - x^5 - 4x^3 + \dots$          | $\{2^*, 3^4\}$    | Non-orientable    |
| $7_6$      | abCd      | 3.506068...  | $x^4 - 3x^3 - x^2 + \dots$          | $\{4^*, 3^2\}$    | Non-orientable    |
| $7_7$      | abCF      | 2.965572...  | $x^4 - 3x^3 + x^2 + \dots$          | $\{4^*, 3^2\}$    | Non-orientable    |
| $8_{12}$   | aBcD      | 4.390256...  | $x^4 - 7x^3 + 13x^2 + \dots$        | $\{6^*\}$         | Orientable        |
| $8_{20}$   | abCDf     | 1.684910...  | $x^8 - x^7 - x^6 + \dots$           | $\{1^*, 3^5\}$    | Non-orientable    |
| $8_{21}$   | abCDF     | 2.618033...  | $x^2 - 3x + \dots$                  | $\{2^*, 4^2\}$    | Orientable        |
| $9_{42}$   | abcdF     | 1.963553...  | $x^6 - 2x^5 - x^4 + 3x^3 + \dots$   | $\{1^*, 3^5\}$    | Non-orientable    |
| $9_{44}$   | abCDF     | 2.794972...  | $x^6 - 2x^5 - 6x^3 + \dots$         | $\{2^*, 3^4\}$    | Non-orientable    |
| $9_{45}$   | aBcdF     | 4.254228...  | $x^6 - 6x^5 + 8x^4 - 4x^3 + \dots$  | $\{3^*, 3^3\}$    | Non-orientable    |
| $9_{48}$   | aaBAcf    | 5.106964...  | $x^4 - 7x^3 + 11x^2 + \dots$        | $\{6^*\}$         | Orientable        |
| $10_{132}$ | abaCDf    | 1.722083...  | $x^4 - x^3 - x^2 + \dots$           | $\{1^*, 3, 4^2\}$ | Non-orientable    |
| $10_{133}$ | abcDDF    | 3.316512...  | $x^4 - 5x^3 + 7x^2 + \dots$         | $\{2^*, 4^2\}$    | Orientable        |
| $10_{136}$ | aBcdf     | 3.254263...  | $x^4 - 3x^3 + \dots$                | $\{2^*, 3^4\}$    | Non-orientable    |
| $10_{137}$ | abCdF     | 3.732050...  | $x^2 - 4x + \dots$                  | $\{3^*, 3^3\}$    | Non-orientable    |
| $10_{140}$ | abbCDf    | 1.883203...  | $x^4 - 2x^3 + x^2 + \dots$          | $\{1^*, 3^5\}$    | Non-orientable    |
| $10_{145}$ | aabAcf    | 2.369205...  | $x^4 - x^3 - 3x^2 + \dots$          | $\{6^*\}$         | Orientable        |
| $11_{n1}$  | aBcdFF    | 5.143081...  | $x^6 - 7x^5 + 10x^4 - 4x^3 + \dots$ | $\{3^*, 3^3\}$    | Non-orientable    |



Table C.2 – continued from previous page

|         |            |             |                                     |                   |                |
|---------|------------|-------------|-------------------------------------|-------------------|----------------|
| 11n12   | abCDCF     | 2.965572... | $x^4 - 3x^3 + x^2 + \dots$          | $\{2^*, 3^2, 4\}$ | Non-orientable |
| 11n28   | abCDFF     | 3.624685... | $x^6 - 3x^5 - 8x^3 + \dots$         | $\{2^*, 3^4\}$    | Non-orientable |
| 11n38   | abacdF     | 1.916498... | $x^8 - x^7 - x^6 - x^5 + \dots$     | $\{1^*, 3^5\}$    | Non-orientable |
| 11n49   | abbcdF     | 2.369205... | $x^4 - x^3 - 3x^2 + \dots$          | $\{1^*, 3^5\}$    | Non-orientable |
| 11n74   | aaBACBaadf | 4.130159... | $x^4 - 3x^3 - 4x^2 + \dots$         | $\{2^*, 3^4\}$    | Non-orientable |
| 11n91   | aBBcdF     | 6.165847... | $x^6 - 8x^5 + 12x^4 - 6x^3 + \dots$ | $\{3^*, 3^3\}$    | Non-orientable |
| 11n102  | aabACf     | 2.751464... | $x^6 - 3x^5 + 2x^4 - 4x^3 + \dots$  | $\{2^*, 3^4\}$    | Non-orientable |
| 11n113  | aBcBdF     | 7.167894... | $x^6 - 9x^5 + 14x^4 - 8x^3 + \dots$ | $\{3^*, 3^3\}$    | Non-orientable |
| 11n116  | abAbcdF    | 2.618033... | $x^2 - 3x + \dots$                  | $\{1^*, 3^5\}$    | Non-orientable |
| 11n142  | aaBACDf    | 5.551933... | $x^4 - 8x^3 + 15x^2 + \dots$        | $\{4^*, 4\}$      | Orientable     |
| 12n0013 | abCddF     | 4.713255... | $x^6 - 7x^5 + 12x^4 - 8x^3 + \dots$ | $\{3^*, 3^3\}$    | Non-orientable |
| 12n0025 | aBacdF     | 3.506068... | $x^4 - 3x^3 - x^2 + \dots$          | $\{2^*, 3^2, 4\}$ | Non-orientable |
| 12n0079 | abcDDDF    | 4.174673... | $x^4 - 6x^3 + 9x^2 + \dots$         | $\{2^*, 4^2\}$    | Orientable     |
| 12n0121 | aabaCDf    | 1.883203... | $x^4 - 2x^3 + x^2 + \dots$          | $\{1^*, 7\}$      | Non-orientable |
| 12n0145 | abbCdF     | 4.525603... | $x^6 - 6x^5 + 6x^4 + 2x^3 + \dots$  | $\{3^*, 3^3\}$    | Non-orientable |
| 12n0282 | abaCdF     | 4.390256... | $x^4 - 7x^3 + 13x^2 + \dots$        | $\{3^*, 3, 4\}$   | Non-orientable |
| 12n0462 | aaBACBdF   | 4.502476... | $x^6 - 4x^5 - 10x^3 + \dots$        | $\{2^*, 3^4\}$    | Non-orientable |
| 12n0582 | abbbCDf    | 2.018790... | $x^6 - x^5 - 2x^4 + x^3 + \dots$    | $\{1^*, 3^5\}$    | Non-orientable |
| 12n0642 | aabbCBaf   | 8.794621... | $x^4 - 7x^3 - 15x^2 + \dots$        | $\{6^*\}$         | Orientable     |
| 12n0838 | aabacBAdCF | 4.330640... | $x^4 - 5x^3 + 4x^2 + \dots$         | $\{2^*, 3^4\}$    | Non-orientable |

Table C.3: Fibred knot complements with fibre  $S_{3,1}$ .

| Name            | Monodromy | $\lambda(h)$ | $\mu_{\lambda(h)}(x)$                               | Stratum             | Stable lamination |
|-----------------|-----------|--------------|---|---------------------|-------------------|
| 8 <sub>2</sub>  | abcdeF    | 2.042490...  | $x^6 - 3x^5 + 3x^4 - 3x^3 + \dots$                  | $\{2^*, 6^2\}$      | Orientable        |
| 8 <sub>5</sub>  | abcdeH    | 2.296630...  | $x^4 - 2x^3 + \dots$                                | $\{2^*, 3^8\}$      | Non-orientable    |
| 8 <sub>7</sub>  | abcdEF    | 1.809789...  | $x^8 - x^7 - 2x^5 + \dots$                          | $\{2^*, 3^2, 5^2\}$ | Non-orientable    |
| 8 <sub>9</sub>  | abcDEF    | 1.754877...  | $x^3 - 2x^2 + x - 1$                                | $\{2^*, 4^4\}$      | Orientable        |
| 8 <sub>10</sub> | abcDEh    | 2.113831...  | $x^{10} - 3x^9 + 3x^8 - 4x^7 + 5x^6 - 5x^5 + \dots$ | $\{2^*, 3^8\}$      | Non-orientable    |

Table C.3 – continued from previous page

|                  |              |             |   |                     |                |
|------------------|--------------|-------------|---|---------------------|----------------|
| 8 <sub>16</sub>  | abcBdCeH     | 2.463632... | $x^{12} - 3x^{11} + x^{10} + 2x^9 - 4x^8 + 2x^7 + x^6 + \dots$  | $\{3^*, 3^7\}$      | Non-orientable |
| 8 <sub>17</sub>  | abcBDCEH     | 2.348693... | $x^{12} - 3x^{11} + x^{10} + 2x^9 - 2x^8 - 2x^7 + 7x^6 + \dots$ | $\{3^*, 3^7\}$      | Non-orientable |
| 8 <sub>18</sub>  | abAcBdEDCHcd | 2.618033... | $x^2 - 3x + \dots$  | $\{4^*, 4^3\}$      | Orientable     |
| 9 <sub>11</sub>  | abcdEf       | 3.422275... | $x^6 - 3x^5 - x^4 - x^3 + \dots$                                | $\{4^*, 5^2\}$      | Non-orientable |
| 9 <sub>17</sub>  | aBcEgHCDeF   | 2.421229... | $x^6 - 3x^5 + x^4 + x^3 + \dots$                                | $\{4^*, 5^2\}$      | Non-orientable |
| 9 <sub>20</sub>  | abcDef       | 3.176002... | $x^6 - 3x^5 + x^4 - 5x^3 + \dots$                               | $\{4^*, 3^2, 4^2\}$ | Non-orientable |
| 9 <sub>22</sub>  | abcdEH       | 2.848930... | $x^8 - 2x^7 - 4x^6 + 2x^5 + 7x^4 + \dots$                       | $\{4^*, 3^6\}$      | Non-orientable |
| 9 <sub>24</sub>  | abcDEH       | 2.865318... | $x^6 - 3x^5 + 2x^4 - 5x^3 + \dots$                              | $\{4^*, 3^2, 4^2\}$ | Non-orientable |
| 9 <sub>26</sub>  | abcDEf       | 2.738661... | $x^6 - 3x^5 + 3x^4 - 7x^3 + \dots$                              | $\{4^*, 3^2, 4^2\}$ | Non-orientable |
| 9 <sub>27</sub>  | abCDEf       | 2.475412... | $x^6 - 3x^5 + 3x^4 - 5x^3 + \dots$                              | $\{4^*, 3^2, 4^2\}$ | Non-orientable |
| 9 <sub>28</sub>  | abCdeH       | 2.794972... | $x^6 - 2x^5 - 6x^3 + \dots$                                     | $\{4^*, 3^6\}$      | Non-orientable |
| 9 <sub>29</sub>  | abAbCDEH     | 3.510048... | $x^{10} - 3x^9 - 3x^8 + 4x^7 + x^6 - x^5 + \dots$               | $\{4^*, 3^6\}$      | Non-orientable |
| 9 <sub>30</sub>  | abCdeH       | 2.738923... | $x^{10} - 3x^9 + x^7 + x^6 + 4x^5 + \dots$                      | $\{4^*, 3^6\}$      | Non-orientable |
| 9 <sub>31</sub>  | abCDef       | 2.595133... | $x^8 - x^7 - 8x^5 - 4x^4 + \dots$                               | $\{4^*, 3^6\}$      | Non-orientable |
| 9 <sub>32</sub>  | abcBDCeH     | 3.289320... | $x^{10} - 5x^9 + 7x^8 - 4x^7 - 4x^6 + 9x^5 + \dots$             | $\{5^*, 3^5\}$      | Non-orientable |
| 9 <sub>33</sub>  | abcBdCEH     | 3.209989... | $x^{10} - 5x^9 + 7x^8 - 4x^7 - 2x^6 + 7x^5 + \dots$             | $\{5^*, 3^5\}$      | Non-orientable |
| 9 <sub>34</sub>  | aabAcBeHCD   | 3.376441... | $x^{10} - 5x^9 + 7x^8 - 6x^7 + 2x^6 + 3x^5 + \dots$             | $\{5^*, 3^5\}$      | Non-orientable |
| 9 <sub>36</sub>  | abcDeh       | 3.664173... | $x^{10} - 3x^9 - 3x^8 + 2x^7 + x^6 - 3x^5 + \dots$              | $\{4^*, 3^6\}$      | Non-orientable |
| 9 <sub>40</sub>  | abacbHCBDCed | 3.732050... | $x^2 - 4x + \dots$  | $\{6^*, 3^4\}$      | Non-orientable |
| 9 <sub>43</sub>  | abcdEh       | 2.225867... | $x^6 - 3x^5 + 2x^4 - x^3 + \dots$                               | $\{2^*, 4^4\}$      | Orientable     |
| 9 <sub>47</sub>  | aabAcBEHCD   | 2.296630... | $x^4 - 2x^3 + \dots$  | $\{3^*, 3^7\}$      | Non-orientable |
| 10 <sub>29</sub> | abcDeF       | 4.223748... | $x^6 - 7x^5 + 15x^4 - 17x^3 + \dots$                            | $\{6^*, 4^2\}$      | Orientable     |
| 10 <sub>41</sub> | aBcdeF       | 3.517657... | $x^6 - 7x^5 + 17x^4 - 21x^3 + \dots$                            | $\{6^*, 4^2\}$      | Orientable     |
| 10 <sub>42</sub> | abCDeF       | 3.673707... | $x^8 - 5x^7 + 6x^6 - 8x^5 + 16x^4 + \dots$                      | $\{6^*, 3^4\}$      | Non-orientable |
| 10 <sub>43</sub> | abCdEF       | 4.099915... | $x^8 - 5x^7 + 4x^6 - 4x^5 + 12x^4 + \dots$                      | $\{6^*, 3^4\}$      | Non-orientable |
| 10 <sub>44</sub> | abCdeF       | 3.545348... | $x^8 - 5x^7 + 6x^6 - 6x^5 + 12x^4 + \dots$                      | $\{6^*, 3^4\}$      | Non-orientable |
| 10 <sub>45</sub> | aBCdeF       | 3.108681... | $x^8 - 5x^7 + 8x^6 - 12x^5 + 20x^4 + \dots$                     | $\{6^*, 3^4\}$      | Non-orientable |
| 10 <sub>59</sub> | aBcdEh       | 3.860908... | $x^8 - 5x^7 + 3x^6 + 6x^5 - 4x^4 + \dots$                       | $\{6^*, 3^4\}$      | Non-orientable |
| 10 <sub>60</sub> | aBCDeh       | 3.294168... | $x^6 - 4x^5 + 8x^3 + \dots$                                     | $\{6^*, 3^4\}$      | Non-orientable |
| 10 <sub>69</sub> | abAbCDef     | 3.583704... | $x^8 - 5x^7 + 8x^6 - 14x^5 + 16x^4 + \dots$                     | $\{6^*, 3^4\}$      | Non-orientable |

Table C.3 – continued from previous page

|                   |               |             |  |                     |                |
|-------------------|---------------|-------------|--|---------------------|----------------|
| 10 <sub>70</sub>  | abcDeH        | 3.914634... | $x^6 - 7x^5 + 16x^4 - 19x^3 + \dots$                               | $\{6^*, 4^2\}$      | Orientable     |
| 10 <sub>71</sub>  | abCdEH        | 3.897414... | $x^8 - 5x^7 + 5x^6 - 6x^5 + 14x^4 + \dots$                         | $\{6^*, 3^4\}$      | Non-orientable |
| 10 <sub>73</sub>  | abCDeh        | 3.566972... | $x^6 - 3x^5 - 7x^3 + \dots$  | $\{6^*, 3^4\}$      | Non-orientable |
| 10 <sub>75</sub>  | abAbCDEf      | 3.147899... | $x^3 - 4x^2 + 3x - 1$  | $\{6^*, 4^2\}$      | Orientable     |
| 10 <sub>78</sub>  | abCdeh        | 4.419480... | $x^4 - 4x^3 - x^2 + \dots$   | $\{6^*, 3^4\}$      | Non-orientable |
| 10 <sub>81</sub>  | abcBDcEH      | 5.207318... | $x^6 - 4x^5 - 4x^4 - 11x^3 + \dots$                                | $\{6^*, 3^4\}$      | Non-orientable |
| 10 <sub>88</sub>  | abCBdcEH      | 4.025599... | $x^8 - 7x^7 + 17x^6 - 27x^5 + 33x^4 + \dots$                       | $\{7^*, 3^3\}$      | Non-orientable |
| 10 <sub>89</sub>  | abCBehcD      | 4.085277... | $x^6 - 6x^5 + 10x^4 - 11x^3 + \dots$                               | $\{7^*, 3^3\}$      | Non-orientable |
| 10 <sub>96</sub>  | abAbCDeH      | 4.062489... | $x^8 - 5x^7 + 3x^6 + 4x^5 - 4x^4 + \dots$                          | $\{6^*, 3^4\}$      | Non-orientable |
| 10 <sub>105</sub> | abcBeHcd      | 4.825711... | $x^6 - 4x^5 - 2x^4 - 9x^3 + \dots$                                 | $\{6^*, 3^4\}$      | Non-orientable |
| 10 <sub>107</sub> | abcBDCEh      | 4.892142... | $x^6 - 4x^5 - 2x^4 - 11x^3 + \dots$                                | $\{6^*, 3^4\}$      | Non-orientable |
| 10 <sub>110</sub> | abcDehCD      | 5.178144... | $x^8 - 6x^7 + 3x^6 + 7x^5 - 4x^4 + \dots$                          | $\{6^*, 3^4\}$      | Non-orientable |
| 10 <sub>115</sub> | abAcbdEDcHCD  | 5.453134... | $x^8 - 8x^7 + 18x^6 - 28x^5 + 35x^4 + \dots$                       | $\{7^*, 3^3\}$      | Non-orientable |
| 10 <sub>125</sub> | abcdEGh       | 1.532925... | $x^{10} - x^9 - x^8 + x^5 + \dots$                                 | $\{1^*, 3^5, 4^2\}$ | Non-orientable |
| 10 <sub>126</sub> | abcDEGh       | 1.696851... | $x^{10} - x^9 - 2x^8 + 2x^7 - 2x^5 + \dots$                        | $\{1^*, 3^9\}$      | Non-orientable |
| 10 <sub>127</sub> | abcdeGH       | 2.575954... | $x^8 - 2x^7 - x^6 - x^5 + \dots$                                   | $\{2^*, 3^8\}$      | Non-orientable |
| 10 <sub>138</sub> | abCDeH        | 2.618033... | $x^2 - 3x + \dots$   | $\{4^*, 4^3\}$      | Orientable     |
| 10 <sub>141</sub> | abcBdehCDEG   | 1.883203... | $x^4 - 2x^3 + x^2 + \dots$   | $\{2^*, 6^2\}$      | Orientable     |
| 10 <sub>143</sub> | abcbdcEDCGH   | 2.121008... | $x^{12} - x^{11} - 3x^{10} + 3x^8 + x^7 - 4x^6 + \dots$            | $\{2^*, 3^8\}$      | Non-orientable |
| 10 <sub>148</sub> | abcBEGhcD     | 2.369205... | $x^4 - x^3 - 3x^2 + \dots$   | $\{2^*, 3^8\}$      | Non-orientable |
| 10 <sub>149</sub> | abcBdceGH     | 3.343594... | $x^{12} - 4x^{11} + x^{10} + 4x^9 + x^8 - 4x^7 + 3x^6 + \dots$     | $\{3^*, 3^7\}$      | Non-orientable |
| 10 <sub>150</sub> | abcbdcCeH     | 3.064264... | $x^{12} - 3x^{11} - x^{10} + 2x^9 + 2x^8 - 2x^7 + x^6 + \dots$     | $\{3^*, 3^7\}$      | Non-orientable |
| 10 <sub>151</sub> | abcBDCEH      | 2.806933... | $x^{10} - x^9 - 3x^8 - 4x^7 - 4x^6 - x^5 + \dots$                  | $\{3^*, 3^7\}$      | Non-orientable |
| 10 <sub>153</sub> | abcBEGhcD     | 1.752377... | $x^{14} - 2x^{13} + x^{12} - x^{11} + x^{10} - x^9 - 2x^7 + \dots$ | $\{1^*, 3^9\}$      | Non-orientable |
| 10 <sub>154</sub> | abcBehcd      | 2.675099... | $x^6 - 4x^4 - 7x^3 + \dots$  | $\{10^*\}$          | Orientable     |
| 10 <sub>155</sub> | abcbDCEGh     | 1.963553... | $x^6 - 2x^5 - x^4 + 3x^3 + \dots$                                  | $\{2^*, 3^8\}$      | Non-orientable |
| 10 <sub>156</sub> | abcbDCEh      | 2.445733... | $x^{10} - x^9 - x^8 - 4x^7 - 4x^6 - x^5 + \dots$                   | $\{3^*, 3^7\}$      | Non-orientable |
| 10 <sub>157</sub> | abCDCeGhcBdce | 3.855540... | $x^6 - 6x^5 + 11x^4 - 13x^3 + \dots$                               | $\{4^*, 4^3\}$      | Orientable     |
| 10 <sub>158</sub> | aabAcBDCEH    | 2.641923... | $x^{10} - x^9 - 4x^8 - 6x^5 + \dots$                               | $\{3^*, 3^7\}$      | Non-orientable |
| 10 <sub>159</sub> | abacBDCEGHcdE | 2.571700... | $x^{12} - 3x^{11} + 3x^9 - 3x^6 + \dots$                           | $\{3^*, 3^7\}$      | Non-orientable |

Table C.3 – continued from previous page

|                   |           |             |   |                |                |
|-------------------|-----------|-------------|---|----------------|----------------|
| 10 <sub>160</sub> | abcBdCeh  | 2.867469... | $x^{10} - 2x^9 - 2x^8 - 3x^7 + 3x^6 + 4x^5 + \dots$               | $\{3^*, 3^7\}$ | Non-orientable |
| 10 <sub>161</sub> | abcbdCeh  | 1.987793... | $x^6 - 2x^4 - 3x^3 + \dots$                                       | $\{10^*\}$     | Orientable     |
| 11a5              | aBcdEH    | 4.280520... | $x^6 - 7x^5 + 14x^4 - 13x^3 + \dots$                              | $\{8^*, 3^2\}$ | Non-orientable |
| 11a17             | abcBDceH  | 6.128475... | $x^6 - 8x^5 + 12x^4 - 5x^3 + \dots$                               | $\{8^*, 3^2\}$ | Non-orientable |
| 11a42             | abCdEh    | 5.082706... | $x^6 - 7x^5 + 10x^4 - 3x^3 + \dots$                               | $\{8^*, 3^2\}$ | Non-orientable |
| 11a51             | aBcDeh    | 4.419480... | $x^4 - 4x^3 - x^2 + \dots$  | $\{8^*, 3^2\}$ | Non-orientable |
| 11a96             | aBcDEf    | 4.345658... | $x^6 - 7x^5 + 13x^4 - 9x^3 + \dots$                               | $\{8^*, 3^2\}$ | Non-orientable |
| 11a121            | aBcdEf    | 4.037462... | $x^6 - 7x^5 + 13x^4 - 7x^3 + \dots$                               | $\{8^*, 3^2\}$ | Non-orientable |
| 11a128            | abAbCDeF  | 4.223748... | $x^6 - 7x^5 + 15x^4 - 17x^3 + \dots$                              | $\{8^*, 3^2\}$ | Non-orientable |
| 11a159            | abCdEf    | 4.706622... | $x^6 - 7x^5 + 11x^4 - 3x^3 + \dots$                               | $\{8^*, 3^2\}$ | Non-orientable |
| 11a209            | aBcBdCEh  | 5.498599... | $x^6 - 8x^5 + 16x^4 - 15x^3 + \dots$                              | $\{8^*, 3^2\}$ | Non-orientable |
| 11a218            | abcBDCeh  | 5.782742... | $x^6 - 8x^5 + 14x^4 - 9x^3 + \dots$                               | $\{8^*, 3^2\}$ | Non-orientable |
| 11a228            | abcBEHCd  | 5.872394... | $x^6 - 8x^5 + 14x^4 - 11x^3 + \dots$                              | $\{8^*, 3^2\}$ | Non-orientable |
| 11n4              | abCBdCegH | 3.096858... | $x^{10} - 3x^9 - 2x^8 + 6x^7 + x^6 - 11x^5 + \dots$               | $\{4^*, 3^6\}$ | Non-orientable |
| 11n5              | abCBEghCD | 3.720776... | $x^{10} - 6x^9 + 9x^8 - 9x^6 + 9x^5 + \dots$                      | $\{5^*, 3^5\}$ | Non-orientable |
| 11n6              | abCBDCEgH | 2.899489... | $x^{12} - 4x^{11} + 4x^{10} - 4x^9 + 6x^8 - 7x^7 + 12x^6 + \dots$ | $\{3^*, 3^7\}$ | Non-orientable |
| 11n7              | abcBdcEH  | 3.916099... | $x^{10} - 5x^9 + 3x^8 + 6x^7 - 4x^6 - x^5 + \dots$                | $\{5^*, 3^5\}$ | Non-orientable |
| 11n8              | abcBehcD  | 4.513209... | $x^{10} - 5x^9 + x^8 + 6x^7 - 2x^6 - 3x^5 + \dots$                | $\{5^*, 3^5\}$ | Non-orientable |
| 11n10             | abcBeGhcD | 5.017079... | $x^{10} - 6x^9 + 3x^8 + 10x^7 - x^6 - 3x^5 + \dots$               | $\{4^*, 3^6\}$ | Non-orientable |
| 11n11             | abcBdcEGH | 3.783227... | $x^{12} - 4x^{11} + 2x^9 + 4x^8 + 3x^7 - 10x^6 + \dots$           | $\{3^*, 3^7\}$ | Non-orientable |
| 11n14             | abcDeGh   | 4.355987... | $x^{12} - 6x^{11} + 6x^{10} + 7x^9 - 8x^8 - 4x^7 + 10x^6 + \dots$ | $\{3^*, 3^7\}$ | Non-orientable |
| 11n15             | abcdEGH   | 2.727069... | $x^6 - 2x^5 - 2x^4 + x^3 + \dots$                                 | $\{2^*, 3^8\}$ | Non-orientable |
| 11n19             | abcdeGh   | 1.839253... | $x^{14} - x^{13} - x^{12} - 2x^{10} + x^8 - x^7 + \dots$          | $\{1^*, 3^9\}$ | Non-orientable |
| 11n21             | abcbDCEgH | 2.853787... | $x^{12} - 4x^{11} + 3x^{10} + 2x^9 - 3x^8 - 4x^7 + 9x^6 + \dots$  | $\{3^*, 3^7\}$ | Non-orientable |
| 11n22             | abcBDCEgH | 3.159689... | $x^8 - 3x^7 - 2x^5 + 2x^4 + \dots$                                | $\{3^*, 3^7\}$ | Non-orientable |
| 11n24             | abcBdCegH | 2.276705... | $x^8 - 3x^7 + 2x^6 - 2x^4 + \dots$                                | $\{2^*, 3^8\}$ | Non-orientable |
| 11n25             | abcBdCeGH | 2.861684... | $x^{12} - 4x^{11} + 3x^{10} + 2x^9 - 5x^8 + 4x^7 - x^6 + \dots$   | $\{3^*, 3^7\}$ | Non-orientable |
| 11n26             | abcbdCEGH | 3.333734... | $x^{12} - 4x^{11} + 2x^{10} + 2x^8 + 3x^7 - 6x^6 + \dots$         | $\{3^*, 3^7\}$ | Non-orientable |
| 11n31             | abcBEhcd  | 2.687180... | $x^{12} - 2x^{11} - x^{10} - x^9 - x^8 - 5x^7 - 2x^6 + \dots$     | $\{2^*, 3^8\}$ | Non-orientable |
| 11n32             | abcBDCEH  | 3.322457... | $x^8 - 3x^7 - x^6 - x^5 + 3x^4 + \dots$                           | $\{5^*, 3^5\}$ | Non-orientable |

Table C.3 – continued from previous page

|        |               |             |  |                     |                |
|--------|---------------|-------------|--|---------------------|----------------|
| 11n33  | abcdbdCEH     | 3.477581... | $x^{10} - 5x^9 + 5x^8 + 2x^7 - 4x^6 + 3x^5 + \dots$                    | $\{5^*, 3^5\}$      | Non-orientable |
| 11n37  | abcdbDCEGH    | 2.118091... | $x^{10} - 3x^9 + 3x^8 - 3x^7 + 2x^6 - 2x^5 + \dots$                    | $\{2^*, 3^8\}$      | Non-orientable |
| 11n48  | abcBdeHCDEG   | 2.124860... | $x^8 - x^7 - x^6 - 6x^4 + \dots$                                       | $\{2^*, 3^8\}$      | Non-orientable |
| 11n51  | aBCDegH       | 2.444771... | $x^8 - 3x^7 + x^6 - x^5 + 5x^4 + \dots$                                | $\{3^*, 3^3, 4^2\}$ | Non-orientable |
| 11n52  | aBCDEgh       | 3.013560... | $x^{10} - 4x^9 + x^8 + 7x^7 - x^6 - 7x^5 + \dots$                      | $\{4^*, 3^6\}$      | Non-orientable |
| 11n53  | aBCdegH       | 2.591940... | $x^{12} - 3x^{11} - x^{10} + 7x^9 - 4x^8 - 4x^7 + 10x^6 + \dots$       | $\{3^*, 3^7\}$      | Non-orientable |
| 11n54  | abCDegH       | 2.575954... | $x^8 - 2x^7 - x^6 - x^5 + \dots$                                       | $\{3^*, 3^7\}$      | Non-orientable |
| 11n55  | abCDEgh       | 3.142262... | $x^6 - 4x^5 + 4x^4 - 5x^3 + \dots$                                     | $\{4^*, 3^2, 4^2\}$ | Non-orientable |
| 11n56  | abCdegH       | 2.637916... | $x^{10} - x^9 - 2x^8 - 4x^7 - 4x^6 - 2x^5 + \dots$                     | $\{3^*, 3^7\}$      | Non-orientable |
| 11n58  | abcDEgh       | 2.894397... | $x^{10} - 4x^9 + 5x^8 - 7x^7 + 7x^6 - 7x^5 + \dots$                    | $\{2^*, 3^8\}$      | Non-orientable |
| 11n59  | abcDegH       | 3.732050... | $x^2 - 4x + \dots$   | $\{3^*, 3^3, 4^2\}$ | Non-orientable |
| 11n66  | abAbCdeH      | 4.142532... | $x^8 - 5x^7 + 5x^6 - 8x^5 + 10x^4 + \dots$                             | $\{6^*, 3^4\}$      | Non-orientable |
| 11n70  | abcdegH       | 2.192319... | $x^{14} - x^{13} - 3x^{12} + 2x^{10} - x^8 + x^7 + \dots$              | $\{1^*, 3^9\}$      | Non-orientable |
| 11n80  | abcBeGhcd     | 2.644612... | $x^{14} - 2x^{13} - 4x^{12} + 4x^{11} + 6x^{10} - 4x^8 + 2x^7 + \dots$ | $\{1^*, 3^9\}$      | Non-orientable |
| 11n82  | abcdbcEDCGh   | 2.081018... | $x^4 - x^3 - 2x^2 + \dots$   | $\{2^*, 3^4, 4^2\}$ | Non-orientable |
| 11n85  | abcDCBedCgH   | 2.919248... | $x^6 - 3x^5 + x^3 + \dots$   | $\{4^*, 5^2\}$      | Non-orientable |
| 11n86  | abAcBdEDCGHcd | 2.402072... | $x^{12} - 3x^{11} + 2x^{10} - 2x^9 + 2x^8 - x^7 + \dots$               | $\{3^*, 3^7\}$      | Non-orientable |
| 11n87  | abCBdcEDgeH   | 3.130001... | $x^4 - 2x^3 - 3x^2 + \dots$  | $\{4^*, 5^2\}$      | Non-orientable |
| 11n89  | abcBdceDgEH   | 4.846774... | $x^{10} - 5x^9 - x^8 + 8x^7 + 3x^6 - 5x^5 + \dots$                     | $\{4^*, 3^6\}$      | Non-orientable |
| 11n92  | abcdGhCdE     | 2.056064... | $x^{12} - x^{11} - 2x^9 - x^8 - 5x^7 - 4x^6 + \dots$                   | $\{2^*, 3^8\}$      | Non-orientable |
| 11n94  | abAcBdEDCGHcd | 3.316512... | $x^4 - 5x^3 + 7x^2 + \dots$  | $\{4^*, 4^3\}$      | Orientable     |
| 11n95  | abAcbdCeDGh   | 3.413942... | $x^8 - x^7 - 4x^6 - 10x^5 - 12x^4 + \dots$                             | $\{3^*, 3^7\}$      | Non-orientable |
| 11n96  | abacEHCBDCEdG | 2.153721... | $x^4 - 3x^3 + 3x^2 + \dots$  | $\{2^*, 3^2, 4^3\}$ | Non-orientable |
| 11n98  | abAbCDEh      | 4.174673... | $x^4 - 6x^3 + 9x^2 + \dots$  | $\{6^*, 4^2\}$      | Orientable     |
| 11n103 | abcBcDef      | 4.685441... | $x^8 - 5x^7 + 2x^6 - 4x^5 + 8x^4 + \dots$                              | $\{6^*, 3^4\}$      | Non-orientable |
| 11n105 | abcBdcEDgeH   | 4.419480... | $x^4 - 4x^3 - x^2 + \dots$   | $\{4^*, 3^2, 4^2\}$ | Non-orientable |
| 11n106 | abcBdCeDcgH   | 2.286042... | $x^{10} - 3x^9 + 3x^8 - 4x^7 + 3x^6 - 3x^5 + \dots$                    | $\{2^*, 3^8\}$      | Non-orientable |
| 11n108 | abAcbdceHCDeG | 5.556559... | $x^{10} - 7x^9 + 8x^8 + x^7 - 6x^6 + 7x^5 + \dots$                     | $\{5^*, 3^5\}$      | Non-orientable |
| 11n109 | abAcbdCeDgH   | 5.066265... | $x^8 - 4x^7 - 7x^6 + 5x^5 + 15x^4 + \dots$                             | $\{4^*, 3^6\}$      | Non-orientable |
| 11n110 | abacBdEDCGHcd | 2.638448... | $x^{12} - 3x^{11} + 4x^9 - 4x^8 - x^7 + 4x^6 + \dots$                  | $\{3^*, 3^7\}$      | Non-orientable |

Table C.3 – continued from previous page

|        |               |             |   |                     |                |
|--------|---------------|-------------|---|---------------------|----------------|
| 11n111 | abcBdceGh     | 2.072341... | $x^{12} - 3x^{11} + x^{10} + 2x^9 - x^6 + \dots$                  | $\{1^*, 3^9\}$      | Non-orientable |
| 11n112 | abAcbdCeDCCGh | 3.512824... | $x^{12} - 4x^{11} + x^{10} + 4x^9 - 7x^8 + 6x^7 - x^6 + \dots$    | $\{3^*, 3^7\}$      | Non-orientable |
| 11n115 | abcBeHCD      | 3.486572... | $x^6 - 4x^5 + 2x^4 - x^3 + \dots$                                 | $\{5^*, 3^5\}$      | Non-orientable |
| 11n118 | abcdCCeGh     | 2.847506... | $x^6 - 4x^5 + 4x^4 - 3x^3 + \dots$                                | $\{2^*, 4^4\}$      | Orientable     |
| 11n119 | abccDCEgH     | 3.359623... | $x^{10} - 4x^9 + 2x^8 + x^7 - x^6 - 2x^5 + \dots$                 | $\{4^*, 3^6\}$      | Non-orientable |
| 11n121 | abcbDCEh      | 4.196472... | $x^{10} - 5x^9 + 3x^8 + 2x^7 - 2x^6 + x^5 + \dots$                | $\{5^*, 3^5\}$      | Non-orientable |
| 11n124 | abcBdCEh      | 3.519329... | $x^{10} - 5x^9 + 5x^8 + 2x^7 - 6x^6 + 7x^5 + \dots$               | $\{5^*, 3^5\}$      | Non-orientable |
| 11n125 | abacBdcEgHCDE | 3.642607... | $x^{10} - 4x^9 + 2x^8 - x^7 - 7x^6 + 7x^5 + \dots$                | $\{4^*, 3^6\}$      | Non-orientable |
| 11n127 | aBcDegHCd     | 3.282174... | $x^{10} - 3x^9 - 2x^8 + 4x^7 + x^6 - 9x^5 + \dots$                | $\{4^*, 3^6\}$      | Non-orientable |
| 11n128 | abcbdcEDCgH   | 3.285605... | $x^{12} - 4x^{11} + 2x^{10} + 3x^9 - 7x^8 + 2x^7 + 3x^6 + \dots$  | $\{3^*, 3^7\}$      | Non-orientable |
| 11n129 | abcdCEGh      | 3.276285... | $x^{12} - 2x^{11} - 4x^{10} - x^9 + 4x^7 + \dots$                 | $\{2^*, 3^8\}$      | Non-orientable |
| 11n130 | abcBDehCDEG   | 3.332807... | $x^{12} - 4x^{11} + 2x^{10} + x^9 - x^8 - 2x^7 + 9x^6 + \dots$    | $\{3^*, 3^7\}$      | Non-orientable |
| 11n131 | aabAcbdCeH    | 3.595580... | $x^6 - 2x^5 - 4x^4 - 5x^3 + \dots$                                | $\{5^*, 3^5\}$      | Non-orientable |
| 11n135 | abcbDCEh      | 2.478892... | $x^8 - 2x^7 - x^6 - x^5 + 2x^4 + \dots$                           | $\{2^*, 3^4, 4^2\}$ | Non-orientable |
| 11n137 | abAbCdef      | 4.858614... | $x^6 - 7x^5 + 13x^4 - 15x^3 + \dots$                              | $\{6^*, 4^2\}$      | Orientable     |
| 11n143 | abbcBdcccEGh  | 2.039781... | $x^{12} - 2x^{11} + x^{10} - 3x^9 + 3x^8 - 4x^7 + 4x^6 + \dots$   | $\{2^*, 3^8\}$      | Non-orientable |
| 11n144 | abcDegHCd     | 4.659064... | $x^6 - 5x^5 + 3x^4 - 7x^3 + \dots$                                | $\{4^*, 3^2, 4^2\}$ | Non-orientable |
| 11n145 | abcBdcEGh     | 2.156605... | $x^{12} - x^{11} - 3x^{10} - x^9 + 3x^8 + x^7 + 4x^6 + \dots$     | $\{1^*, 3^9\}$      | Non-orientable |
| 11n146 | abcBegHCD     | 3.303317... | $x^{10} - 2x^9 - 2x^8 - 7x^7 - 6x^5 + \dots$                      | $\{3^*, 3^7\}$      | Non-orientable |
| 11n154 | abAcBcDCEgH   | 3.984750... | $x^{10} - 6x^9 + 9x^8 - 2x^7 - 11x^6 + 17x^5 + \dots$             | $\{5^*, 3^5\}$      | Non-orientable |
| 11n156 | abbcHCBDCedH  | 3.700932... | $x^{10} - 6x^9 + 10x^8 - 5x^7 - 3x^6 + 5x^5 + \dots$              | $\{5^*, 3^5\}$      | Non-orientable |
| 11n157 | abAcBdCegHCdE | 4.049012... | $x^{10} - 4x^9 - x^7 + x^6 - x^5 + \dots$                         | $\{4^*, 3^6\}$      | Non-orientable |
| 11n159 | abAcbdCBGhCDE | 4.219345... | $x^{10} - 4x^9 - 2x^8 + 5x^7 + x^6 - 13x^5 + \dots$               | $\{4^*, 3^6\}$      | Non-orientable |
| 11n160 | abAcBeGhCCd   | 3.812002... | $x^{10} - 4x^9 - x^8 + 7x^7 - x^6 - 3x^5 + \dots$                 | $\{4^*, 3^6\}$      | Non-orientable |
| 11n163 | abAbcbDCEh    | 4.589511... | $x^8 - 7x^7 + 15x^6 - 23x^5 + 27x^4 + \dots$                      | $\{7^*, 3^3\}$      | Non-orientable |
| 11n164 | abacbhCBDCed  | 4.102191... | $x^6 - 3x^5 - 4x^4 - x^3 + \dots$                                 | $\{4^*, 3^6\}$      | Non-orientable |
| 11n165 | abacbhCBDCeD  | 4.061725... | $x^8 - 5x^7 + 5x^6 - 6x^5 + 6x^4 + \dots$                         | $\{6^*, 3^4\}$      | Non-orientable |
| 11n167 | abAcbdCEDGh   | 4.213271... | $x^{12} - 4x^{11} - 2x^{10} + 4x^9 + 2x^8 + 5x^7 - 10x^6 + \dots$ | $\{3^*, 3^7\}$      | Non-orientable |
| 11n168 | abAbBDCEH     | 4.413117... | $x^{10} - 5x^9 + x^8 + 8x^7 - 4x^6 - x^5 + \dots$                 | $\{5^*, 3^5\}$      | Non-orientable |
| 11n172 | abcBdCEGh     | 3.370595... | $x^{12} - 4x^{11} + 2x^{10} + 7x^7 - 10x^6 + \dots$               | $\{3^*, 3^7\}$      | Non-orientable |

Table C.3 – continued from previous page

|         |               |             |  |                     |                |
|---------|---------------|-------------|--|---------------------|----------------|
| 11n176  | abAcbdCegHCDE | 3.921325... | $x^8 - 5x^7 + 4x^6 + 2x^5 - 5x^4 + \dots$                          | $\{4^*, 3^6\}$      | Non-orientable |
| 11n179  | abAcBEhCCd    | 4.102191... | $x^6 - 3x^5 - 4x^4 - x^3 + \dots$                                  | $\{6^*, 3^4\}$      | Non-orientable |
| 11n183  | abacbhCBdCed  | 3.546455... | $x^4 - 2x^3 - 5x^2 + \dots$  | $\{10^*\}$          | Orientable     |
| 12a0125 | abCBEhCd      | 6.904075... | $x^6 - 12x^5 + 44x^4 - 67x^3 + \dots$                              | $\{10^*\}$          | Orientable     |
| 12a0181 | aBcDeH        | 5.551933... | $x^4 - 8x^3 + 15x^2 + \dots$                                       | $\{10^*\}$          | Orientable     |
| 12a0477 | aBcDeF        | 5.048917... | $x^3 - 6x^2 + 5x - 1$  | $\{10^*\}$          | Orientable     |
| 12a1124 | abAcBdEDcHCd  | 7.698532... | $x^4 - 10x^3 + 19x^2 + \dots$                                      | $\{10^*\}$          | Orientable     |
| 12n0001 | abcBDCEgH     | 3.074645... | $x^{10} - 2x^9 - 3x^8 - 2x^7 + x^6 + 7x^5 + \dots$                 | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0003 | abcBdCEgH     | 3.950993... | $x^{10} - 6x^9 + 9x^8 - 4x^7 - x^6 + 11x^5 + \dots$                | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0004 | abCEghcD      | 3.510756... | $x^{10} - 6x^9 + 11x^8 - 8x^7 - x^6 + 5x^5 + \dots$                | $\{5^*, 3^5\}$      | Non-orientable |
| 12n0005 | abCBDcegH     | 4.240901... | $x^{10} - 6x^9 + 8x^8 - 3x^7 + 6x^6 - 14x^5 + \dots$               | $\{5^*, 3^5\}$      | Non-orientable |
| 12n0007 | abcbDceGH     | 3.268189... | $x^{12} - 4x^{11} + x^{10} + 4x^9 + 3x^8 - 4x^7 + x^6 + \dots$     | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0009 | abcbDceGH     | 3.999162... | $x^{10} - 6x^9 + 9x^8 - 4x^7 - 3x^6 + 13x^5 + \dots$               | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0010 | abCBdcegH     | 3.163925... | $x^8 - 2x^7 - 3x^6 - x^5 - 3x^4 + \dots$                           | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0014 | abCBeghCD     | 4.215521... | $x^{10} - 6x^9 + 8x^8 - 3x^7 + 8x^6 - 18x^5 + \dots$               | $\{5^*, 3^5\}$      | Non-orientable |
| 12n0015 | abCBdCEgH     | 5.159059... | $x^8 - 8x^7 + 15x^6 + x^5 - 15x^4 + \dots$                         | $\{6^*, 3^4\}$      | Non-orientable |
| 12n0017 | abCBdCEH      | 4.725684... | $x^8 - 7x^7 + 11x^6 + x^5 - 11x^4 + \dots$                         | $\{7^*, 3^3\}$      | Non-orientable |
| 12n0018 | abCBDceH      | 4.335594... | $x^8 - 7x^7 + 13x^6 - 5x^5 - 5x^4 + \dots$                         | $\{7^*, 3^3\}$      | Non-orientable |
| 12n0019 | abCBDCEgH     | 2.848767... | $x^{12} - 4x^{11} + 2x^{10} + 6x^9 - 8x^8 + x^7 + 8x^6 + \dots$    | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0020 | abcbDCEgH     | 3.096806... | $x^{12} - 4x^{11} + 4x^{10} - 4x^9 + 3x^7 - 2x^6 + \dots$          | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0024 | abcBdCEGH     | 3.019641... | $x^{12} - 4x^{11} + 4x^{10} - 4x^9 + 2x^8 + x^7 + 2x^6 + \dots$    | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0035 | aBCDegh       | 3.206833... | $x^{10} - 4x^9 + 2x^8 + x^7 + x^6 + 4x^5 + \dots$                  | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0036 | aBCdEgH       | 4.451288... | $x^{10} - 8x^9 + 18x^8 - 7x^7 - 17x^6 + 24x^5 + \dots$             | $\{5^*, 3^5\}$      | Non-orientable |
| 12n0038 | abCDegh       | 3.445037... | $x^8 - 2x^7 - x^6 - 11x^5 - 6x^4 + \dots$                          | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0039 | abCdEgH       | 4.469420... | $x^8 - 6x^7 + 7x^6 - 5x^5 + 20x^4 + \dots$                         | $\{5^*, 3^5\}$      | Non-orientable |
| 12n0041 | aBCDEgH       | 2.363568... | $x^{12} - 3x^{11} + x^{10} - x^9 + 6x^8 - 2x^7 + \dots$            | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0042 | abcDEgH       | 3.221693... | $x^6 - 6x^5 + 12x^4 - 13x^3 + \dots$                               | $\{3^*, 3^3, 4^2\}$ | Non-orientable |
| 12n0044 | abcdEgH       | 3.610609... | $x^{12} - 6x^{11} + 8x^{10} + 5x^9 - 10x^8 - 2x^7 + 10x^6 + \dots$ | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0045 | abCDEgH       | 2.440094... | $x^{10} - x^9 - 3x^8 - 2x^7 + 5x^5 + \dots$                        | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0052 | abAbCDeh      | 4.791287... | $x^2 - 5x + \dots$   | $\{8^*, 3^2\}$      | Non-orientable |

Table C.3 – continued from previous page

|         |             |             |   |                |                |
|---------|-------------|-------------|---|----------------|----------------|
| 12n0054 | abcDegh     | 3.424229... | $x^{12} - 2x^{11} - 4x^{10} - 3x^9 + \dots$                   | $\{2^*, 3^8\}$ | Non-orientable |
| 12n0069 | abCBEGhcd   | 4.657500... | $x^{10} - 6x^9 + 6x^8 + x^7 + 2x^6 - 6x^5 + \dots$            | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0070 | abCBdceGH   | 3.533088... | $x^{10} - 6x^9 + 11x^8 - 8x^7 - 3x^6 + 11x^5 + \dots$         | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0072 | abcBDcegH   | 6.036370... | $x^8 - 7x^7 + 6x^6 - 4x^5 + 18x^4 + \dots$                    | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0073 | abcBEghcD   | 3.442134... | $x^{10} - 4x^9 + x^8 + 4x^7 - 3x^6 + x^5 + \dots$             | $\{3^*, 3^7\}$ | Non-orientable |
| 12n0076 | abcBdceGGH  | 3.832079... | $x^{12} - 5x^{11} + 3x^{10} + 6x^9 - 6x^7 + 5x^6 + \dots$     | $\{3^*, 3^7\}$ | Non-orientable |
| 12n0097 | abCBEGhcD   | 3.059240... | $x^{10} - 3x^9 - 3x^6 + 5x^5 + \dots$                         | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0101 | abAbcDcEHC  | 8.756911... | $x^6 - 10x^5 + 12x^4 - 11x^3 + \dots$                         | $\{8^*, 3^2\}$ | Non-orientable |
| 12n0102 | abAbcdceHC  | 4.339892... | $x^{10} - 4x^9 - x^8 - 3x^7 + 5x^6 - 5x^5 + \dots$            | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0119 | abcBdcegH   | 2.912293... | $x^{12} - x^{11} - 4x^{10} - 4x^9 - x^8 - x^7 - 2x^6 + \dots$ | $\{2^*, 3^8\}$ | Non-orientable |
| 12n0120 | abcBEGEhcD  | 2.332180... | $x^{10} - 3x^9 + 3x^8 - 4x^7 + 4x^6 - 7x^5 + \dots$           | $\{2^*, 3^8\}$ | Non-orientable |
| 12n0140 | abAbcDCegHC | 6.516491... | $x^{10} - 8x^9 + 10x^8 - 3x^7 + 8x^6 - 18x^5 + \dots$         | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0141 | abAbcdgHCDE | 4.086223... | $x^{10} - 5x^9 + 4x^8 - 2x^7 + 7x^6 - 15x^5 + \dots$          | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0142 | abcDCBedcgH | 3.207862... | $x^{10} - 3x^9 - x^7 - 5x^6 + 6x^5 + \dots$                   | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0143 | abcBdceDgeH | 3.090657... | $x^4 - 4x^3 + 4x^2 + \dots$                                   | $\{2^*, 3^8\}$ | Non-orientable |
| 12n0144 | abcBdCEDgEh | 4.566915... | $x^{10} - 5x^9 + 2x^8 - x^7 + x^6 + 14x^5 + \dots$            | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0148 | abcBcdef    | 3.301490... | $x^6 - x^5 - 5x^4 - 7x^3 + \dots$                             | $\{10^*\}$     | Orientable     |
| 12n0151 | aBCdEGh     | 4.157109... | $x^6 - 6x^5 + 8x^4 - 3x^3 + \dots$                            | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0152 | aBCdeGH     | 3.144141... | $x^{10} - 4x^9 + 2x^8 + x^7 + x^6 + 8x^5 + \dots$             | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0157 | abAbCDegH   | 3.986687... | $x^{10} - 5x^9 + 4x^8 + 2x^6 - 6x^5 + \dots$                  | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0158 | abAbCDEgh   | 4.315072... | $x^6 - 8x^5 + 22x^4 - 31x^3 + \dots$                          | $\{6^*, 4^2\}$ | Orientable     |
| 12n0160 | aBcDEGH     | 3.855267... | $x^{10} - 5x^9 + 4x^8 + 2x^7 - 2x^6 + 2x^5 + \dots$           | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0161 | aBcdeGH     | 3.996778... | $x^6 - 8x^5 + 22x^4 - 29x^3 + \dots$                          | $\{6^*, 4^2\}$ | Orientable     |
| 12n0162 | aBcdEGh     | 3.826541... | $x^{10} - 5x^9 + 4x^8 + 2x^7 - 2x^5 + \dots$                  | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0164 | abCdeGH     | 4.994274... | $x^8 - 6x^7 + 5x^6 - 3x^5 + 16x^4 + \dots$                    | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0165 | abCdeGH     | 3.188184... | $x^{10} - 4x^9 + 4x^8 - 9x^7 + 15x^6 - 6x^5 + \dots$          | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0168 | abcdeGGH    | 3.216783... | $x^{10} - 5x^9 + 6x^8 - x^7 + x^6 - 2x^5 + \dots$             | $\{2^*, 3^8\}$ | Non-orientable |
| 12n0178 | abcBcDEH    | 6.040307... | $x^6 - 5x^5 - 4x^4 - 13x^3 + \dots$                           | $\{6^*, 3^4\}$ | Non-orientable |
| 12n0179 | abcBcdeH    | 3.863929... | $x^{12} - 4x^{11} + x^9 + 5x^8 - 4x^7 + x^6 + \dots$          | $\{3^*, 3^7\}$ | Non-orientable |
| 12n0183 | abcBcDegH   | 7.052660... | $x^8 - 8x^7 + 7x^6 - 5x^5 + 20x^4 + \dots$                    | $\{5^*, 3^5\}$ | Non-orientable |



Table C.3 – continued from previous page

|         |               |             |   |                |                |
|---------|---------------|-------------|---|----------------|----------------|
| 12n0184 | abcBcDEgh     | 4.110207... | $x^{12} - 5x^{11} + 4x^{10} - x^9 - 3x^8 + 7x^7 - 8x^6 + \dots$   | $\{3^*, 3^7\}$ | Non-orientable |
| 12n0198 | aBCDEGh       | 2.777651... | $x^{12} - 3x^{11} - x^{10} + 4x^9 + 2x^8 - 2x^7 + x^6 + \dots$    | $\{3^*, 3^7\}$ | Non-orientable |
| 12n0199 | abcDEDGh      | 1.744363... | $x^{14} - x^{13} - x^{12} - 2x^9 + 2x^7 + \dots$                  | $\{1^*, 3^9\}$ | Non-orientable |
| 12n0204 | abAbCdeh      | 6.133502... | $x^6 - 7x^5 + 6x^4 - 5x^3 + \dots$                                | $\{8^*, 3^2\}$ | Non-orientable |
| 12n0211 | abAbCdegH     | 4.054814... | $x^8 - 4x^7 - x^6 + 3x^5 + \dots$                                 | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0218 | abCDEGh       | 2.736024... | $x^{10} - x^9 - 3x^8 - 4x^7 - 2x^6 + x^5 + \dots$                 | $\{3^*, 3^7\}$ | Non-orientable |
| 12n0230 | abCBDCEh      | 3.161717... | $x^{10} - 4x^9 + 2x^8 + x^7 + 3x^6 + \dots$                       | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0269 | abAbCdEH      | 4.929789... | $x^6 - 7x^5 + 12x^4 - 11x^3 + \dots$                              | $\{8^*, 3^2\}$ | Non-orientable |
| 12n0271 | abcDCBGHcdE   | 3.018779... | $x^{10} - 3x^9 + x^7 - 5x^6 + 6x^5 + \dots$                       | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0272 | abcBdCEDgeH   | 4.173411... | $x^8 - 3x^7 - x^6 - 14x^5 - 6x^4 + \dots$                         | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0273 | abcBdceDgEh   | 2.695439... | $x^8 - 2x^7 - x^6 - 2x^5 + \dots$                                 | $\{2^*, 3^8\}$ | Non-orientable |
| 12n0276 | abAcbdCdef    | 3.767506... | $x^6 - x^5 - 7x^4 - 11x^3 + \dots$                                | $\{10^*\}$     | Orientable     |
| 12n0280 | aBccDCEgH     | 3.221693... | $x^6 - 6x^5 + 12x^4 - 13x^3 + \dots$                              | $\{4^*, 4^3\}$ | Orientable     |
| 12n0281 | aBccDCegH     | 3.858863... | $x^{10} - 4x^9 + x^7 + 3x^6 + 4x^5 + \dots$                       | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0284 | aabAcBegHCD   | 3.168457... | $x^{12} - 4x^{11} + 4x^{10} - 6x^9 + 6x^8 - 5x^7 + 10x^6 + \dots$ | $\{3^*, 3^7\}$ | Non-orientable |
| 12n0285 | aabAcBEGHCD   | 2.828384... | $x^6 - x^5 - 3x^4 - 5x^3 + \dots$                                 | $\{3^*, 3^7\}$ | Non-orientable |
| 12n0290 | abcBdeGHcH    | 5.247002... | $x^{12} - 6x^{11} + 2x^{10} + 11x^9 - 3x^8 - 6x^7 + 5x^6 + \dots$ | $\{3^*, 3^7\}$ | Non-orientable |
| 12n0295 | abcDCBGhCdE   | 5.014914... | $x^6 - 9x^5 + 26x^4 - 35x^3 + \dots$                              | $\{6^*, 4^2\}$ | Orientable     |
| 12n0298 | abAbcbDCEgH   | 3.829767... | $x^{10} - 6x^9 + 11x^8 - 12x^7 + 7x^6 - 3x^5 + \dots$             | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0300 | abcBdeHCDeg   | 4.464765... | $x^{10} - 5x^9 + 2x^8 + x^7 + x^6 + 10x^5 + \dots$                | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0302 | abAbCdeF      | 5.004584... | $x^6 - 7x^5 + 11x^4 - 7x^3 + \dots$                               | $\{8^*, 3^2\}$ | Non-orientable |
| 12n0303 | abcdeGHH      | 3.544599... | $x^{10} - 5x^9 + 5x^8 + x^7 - 2x^6 + 2x^5 + \dots$                | $\{2^*, 3^8\}$ | Non-orientable |
| 12n0309 | abcdEGhh      | 1.661047... | $x^8 - 2x^7 + x^6 - x^5 + x^4 + \dots$                            | $\{1^*, 5^3\}$ | Non-orientable |
| 12n0312 | aBcDegHCD     | 3.128116... | $x^{10} - 3x^9 - 5x^6 + 5x^5 + \dots$                             | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0314 | abacbgHCBDCed | 3.631988... | $x^{10} - 4x^9 + 2x^8 - x^7 - 5x^6 + x^5 + \dots$                 | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0315 | abAcBEGhCCd   | 3.898696... | $x^{10} - 4x^9 + x^7 + x^6 + 4x^5 + \dots$                        | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0316 | abcBdceGHH    | 5.201545... | $x^{12} - 6x^{11} + 3x^{10} + 6x^9 + x^8 - 6x^7 + 5x^6 + \dots$   | $\{3^*, 3^7\}$ | Non-orientable |
| 12n0318 | abcBBEGhcd    | 1.815698... | $x^{12} - 3x^{11} + 3x^{10} - x^9 - x^8 + x^7 - 2x^6 + \dots$     | $\{1^*, 3^9\}$ | Non-orientable |
| 12n0329 | abAbcdeh      | 3.546455... | $x^4 - 2x^3 - 5x^2 + \dots$                                       | $\{10^*\}$     | Orientable     |
| 12n0336 | abcBcdegH     | 3.728549... | $x^{12} - 2x^{11} - 5x^{10} - 5x^9 - x^8 - x^7 - 2x^6 + \dots$    | $\{2^*, 3^8\}$ | Non-orientable |

Table C.3 – continued from previous page

|         |                |             |   |                     |                |
|---------|----------------|-------------|---|---------------------|----------------|
| 12n0337 | abcBcDeF       | 5.381600... | $x^6 - 7x^5 + 9x^4 - 3x^3 + \dots$                                  | $\{8^*, 3^2\}$      | Non-orientable |
| 12n0340 | abcBdeehCDEG   | 2.042490... | $x^6 - 3x^5 + 3x^4 - 3x^3 + \dots$                                  | $\{2^*, 4^4\}$      | Orientable     |
| 12n0347 | abcbdcEDCGHH   | 2.456674... | $x^{12} - x^{11} - 4x^{10} + 3x^8 + x^7 - 6x^6 + \dots$             | $\{2^*, 3^8\}$      | Non-orientable |
| 12n0348 | abcdGHcDe      | 4.789269... | $x^{10} - 5x^9 + x^8 - x^7 + 2x^6 + 14x^5 + \dots$                  | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0349 | abcdhCdE       | 3.288438... | $x^{12} - 3x^{11} - x^{10} - x^9 + 4x^8 - 2x^6 + \dots$             | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0350 | abAcbdCBGHCdE  | 5.562917... | $x^{10} - 7x^9 + 9x^8 - 7x^7 + 9x^6 - 8x^5 + \dots$                 | $\{5^*, 3^5\}$      | Non-orientable |
| 12n0351 | abcBdCeGh      | 3.127830... | $x^{12} - 4x^{11} + 3x^{10} - 3x^9 + 8x^8 - 5x^7 + 4x^6 + \dots$    | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0352 | abCEgHCD       | 2.618033... | $x^2 - 3x + \dots$  | $\{3^*, 3^3, 4^2\}$ | Non-orientable |
| 12n0353 | abAcBcDCEgh    | 5.302264... | $x^6 - 9x^5 + 25x^4 - 33x^3 + \dots$                                | $\{6^*, 4^2\}$      | Orientable     |
| 12n0354 | abAcbdCBGHCDE  | 2.856201... | $x^{12} - 3x^{11} + 3x^9 - 6x^8 + 7x^6 + \dots$                     | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0355 | abacbHCBDCED   | 2.667096... | $x^{10} - 3x^9 + x^8 - x^7 + 3x^6 - 4x^5 + \dots$                   | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0356 | abacbHCBdCED   | 3.169660... | $x^{10} - 3x^9 - x^8 + 2x^7 + x^6 - 9x^5 + \dots$                   | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0357 | abacbhCBDCeD   | 6.002530... | $x^8 - 8x^7 + 12x^6 + 2x^5 - 13x^4 + \dots$                         | $\{7^*, 3^3\}$      | Non-orientable |
| 12n0362 | abAcbdcegHCDE  | 2.650603... | $x^{10} - 4x^9 + 5x^8 - 4x^7 - x^6 + 5x^5 + \dots$                  | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0363 | abAcBdceHCDeG  | 4.938346... | $x^{10} - 7x^9 + 12x^8 - 9x^7 - 2x^6 + 11x^5 + \dots$               | $\{5^*, 3^5\}$      | Non-orientable |
| 12n0366 | abcdhCde       | 4.449156... | $x^6 - 2x^5 - 8x^4 - 11x^3 + \dots$                                 | $\{10^*\}$          | Orientable     |
| 12n0367 | abAbcBDCEh     | 6.338286... | $x^6 - 8x^5 + 12x^4 - 11x^3 + \dots$                                | $\{8^*, 3^2\}$      | Non-orientable |
| 12n0370 | abbcdbccEGh    | 1.974818... | $x^6 - 2x^5 + x^4 - 2x^3 + \dots$                                   | $\{2^*, 3^8\}$      | Non-orientable |
| 12n0371 | aabAcBdCBhcbDE | 2.256455... | $x^6 - 2x^5 - x^3 + \dots$  | $\{2^*, 3^2, 4^3\}$ | Non-orientable |
| 12n0372 | aabacEHCBDcEdG | 2.905984... | $x^{12} - 3x^{11} - x^{10} + 4x^9 - 2x^7 - x^6 + \dots$             | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0373 | abCCDCeGhcBdce | 5.319013... | $x^6 - 8x^5 + 18x^4 - 23x^3 + \dots$                                | $\{4^*, 4^3\}$      | Orientable     |
| 12n0375 | abAcbdcbGHcDe  | 4.456859... | $x^{12} - 6x^{11} + 5x^{10} + 11x^9 - 11x^8 - 6x^7 + 14x^6 + \dots$ | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0377 | abAcBdCegHcdE  | 3.157106... | $x^{12} - 2x^{11} - 2x^{10} - 3x^9 - 5x^8 - 5x^7 - 2x^6 + \dots$    | $\{2^*, 3^8\}$      | Non-orientable |
| 12n0385 | abacbdCBegHCD  | 4.194187... | $x^{10} - 6x^9 + 9x^8 - 6x^7 - x^6 + 5x^5 + \dots$                  | $\{5^*, 3^5\}$      | Non-orientable |
| 12n0388 | abCDcBeDCgH    | 2.934377... | $x^{10} - 3x^9 + x^7 - 3x^6 + 6x^5 + \dots$                         | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0389 | abCBdehCDEG    | 5.551933... | $x^4 - 8x^3 + 15x^2 + \dots$  | $\{6^*, 4^2\}$      | Orientable     |
| 12n0390 | abccDCegH      | 3.942573... | $x^{10} - 4x^9 + 2x^8 - 9x^7 + 9x^6 - 6x^5 + \dots$                 | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0392 | abCBdcEh       | 4.822688... | $x^8 - 7x^7 + 11x^6 - x^5 - 7x^4 + \dots$                           | $\{7^*, 3^3\}$      | Non-orientable |
| 12n0395 | abAbcbDCeh     | 5.731419... | $x^6 - 6x^5 + 9x^3 + \dots$   | $\{7^*, 3^3\}$      | Non-orientable |
| 12n0396 | abCBdcEdgeH    | 4.500427... | $x^{10} - 6x^9 + 8x^8 - 5x^7 - 5x^6 + 11x^5 + \dots$                | $\{5^*, 3^5\}$      | Non-orientable |

Table C.3 – continued from previous page

|         |               |             |   |                |                |
|---------|---------------|-------------|---|----------------|----------------|
| 12n0397 | abcBEgHCD     | 2.915008... | $x^{10} - 2x^9 - x^8 - 4x^7 - 3x^6 + 3x^5 + \dots$                  | $\{3^*, 3^7\}$ | Non-orientable |
| 12n0398 | aabAcbdCeH    | 3.496158... | $x^{12} - 3x^{11} - 3x^{10} + 4x^9 + 2x^8 - 2x^7 + x^6 + \dots$     | $\{3^*, 3^7\}$ | Non-orientable |
| 12n0399 | abAcBdCdEGh   | 4.701374... | $x^{10} - 5x^9 + x^8 + x^7 + 2x^6 + 10x^5 + \dots$                  | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0400 | abAcBDCEGHcde | 4.954350... | $x^{10} - 7x^9 + 12x^8 - 9x^7 - 4x^6 + 15x^5 + \dots$               | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0401 | abcDegHCD     | 4.412135... | $x^8 - 3x^7 - 2x^6 - 16x^5 - 8x^4 + \dots$                          | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0402 | abAbcdef      | 2.693117... | $x^6 - x^5 - 3x^4 - 3x^3 + \dots$                                   | $\{10^*\}$     | Orientable     |
| 12n0403 | abAcbdCeDgh   | 2.503822... | $x^6 - 2x^5 - x^4 + \dots$  | $\{2^*, 3^8\}$ | Non-orientable |
| 12n0407 | abcbdceDCGH   | 3.541541... | $x^{12} - 4x^{11} + 5x^9 + 3x^8 - 2x^7 + 3x^6 + \dots$              | $\{2^*, 3^8\}$ | Non-orientable |
| 12n0408 | abcBdcEDcGH   | 5.890961... | $x^8 - 7x^7 + 7x^6 - 6x^5 + 20x^4 + \dots$                          | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0409 | abcBDGhcDE    | 2.894537... | $x^{12} - x^{11} - 6x^{10} + x^9 + 4x^8 - 4x^7 - 9x^6 + \dots$      | $\{2^*, 3^8\}$ | Non-orientable |
| 12n0420 | aabAcBeGHCD   | 4.085832... | $x^{10} - 6x^9 + 9x^8 - 6x^7 + 3x^6 + 7x^5 + \dots$                 | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0423 | abAbcdCegHC   | 3.515350... | $x^{12} - 4x^{11} + x^{10} + 4x^9 - 5x^8 - 4x^7 + 11x^6 + \dots$    | $\{3^*, 3^7\}$ | Non-orientable |
| 12n0431 | abAcCbDCEgH   | 4.374742... | $x^8 - 6x^7 + 7x^6 + 3x^5 - 12x^4 + \dots$                          | $\{6^*, 3^4\}$ | Non-orientable |
| 12n0433 | abcBdcEh      | 2.564312... | $x^{12} - 2x^{11} - 4x^{10} + 3x^9 + 8x^8 + 2x^7 + \dots$           | $\{2^*, 3^8\}$ | Non-orientable |
| 12n0434 | abAcBdEDCGhcd | 3.742915... | $x^{10} - 4x^9 + x^8 + x^7 - 6x^6 + 8x^5 + \dots$                   | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0435 | abaceHCBDCEDg | 3.642607... | $x^{10} - 4x^9 + 2x^8 - x^7 - 7x^6 + 7x^5 + \dots$                  | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0436 | abacbHCBdceD  | 3.890386... | $x^6 - 3x^5 - 2x^4 - 5x^3 + \dots$                                  | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0437 | abcBcEGhcd    | 3.971378... | $x^{14} - 4x^{13} - x^{11} + 5x^{10} + 2x^9 + 5x^8 - 5x^7 + \dots$  | $\{1^*, 3^9\}$ | Non-orientable |
| 12n0438 | abcbdceDCgH   | 3.242769... | $x^{14} - 2x^{13} - 4x^{12} - x^{11} + 3x^{10} - x^8 + x^7 + \dots$ | $\{1^*, 3^9\}$ | Non-orientable |
| 12n0439 | abcBDEGhcd    | 2.038389... | $x^{12} - 3x^{11} + 2x^{10} + x^9 - 2x^8 - x^7 + 2x^6 + \dots$      | $\{1^*, 3^9\}$ | Non-orientable |
| 12n0441 | abAcdeGHCbCh  | 6.753766... | $x^{10} - 7x^9 + x^8 + 4x^7 + 3x^6 + x^5 + \dots$                   | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0446 | abAcbdcehCDeG | 2.513835... | $x^{12} - 2x^{11} - 3x^{10} + 5x^9 - 3x^8 + 8x^6 + \dots$           | $\{2^*, 3^8\}$ | Non-orientable |
| 12n0449 | abAcBeghCCd   | 3.319246... | $x^{12} - 3x^{11} - 3x^{10} + 6x^9 + 2x^8 - 2x^7 + x^6 + \dots$     | $\{3^*, 3^7\}$ | Non-orientable |
| 12n0450 | abacbGHCBDCed | 4.186752... | $x^8 - 7x^7 + 15x^6 - 17x^5 + 18x^4 + \dots$                        | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0451 | abacbbbeGhcd  | 2.649074... | $x^{12} - x^{11} - 5x^{10} + x^9 + 4x^8 - 3x^7 - 7x^6 + \dots$      | $\{2^*, 3^8\}$ | Non-orientable |
| 12n0454 | abcBEGhcDD    | 3.487791... | $x^{12} - 2x^{11} - 6x^{10} + 2x^9 + 5x^8 - 4x^7 - 11x^6 + \dots$   | $\{2^*, 3^8\}$ | Non-orientable |
| 12n0456 | abcBEGhcdd    | 2.958226... | $x^{14} - 3x^{13} - x^{11} + 4x^{10} + 3x^8 - 9x^7 + \dots$         | $\{1^*, 3^9\}$ | Non-orientable |
| 12n0460 | abAcbdcbgHCDE | 3.785885... | $x^{10} - 4x^9 + 2x^8 - 3x^7 - 7x^6 + 7x^5 + \dots$                 | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0461 | abcBDehCDEg   | 3.803329... | $x^{10} - 6x^9 + 10x^8 - 7x^7 + x^6 + 7x^5 + \dots$                 | $\{4^*, 3^6\}$ | Non-orientable |
| 12n0463 | abcDCBedCGH   | 3.891499... | $x^{10} - 6x^9 + 10x^8 - 7x^7 - 3x^6 + 13x^5 + \dots$               | $\{5^*, 3^5\}$ | Non-orientable |

Table C.3 – continued from previous page

|         |                 |             |   |                     |                |
|---------|-----------------|-------------|---|---------------------|----------------|
| 12n0475 | abccdEGh        | 1.686481... | $x^{12} - 2x^{11} + x^{10} - 2x^8 + 2x^7 - 2x^6 + \dots$                    | $\{1^*, 3^9\}$      | Non-orientable |
| 12n0480 | abCDEGhcBdcE    | 4.222722... | $x^{10} - 4x^9 - 2x^8 + 5x^7 + 3x^6 - 23x^5 + \dots$                        | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0481 | abAcBdCeDCCGh   | 3.721802... | $x^{12} - 4x^{11} - x^{10} + 9x^9 - 4x^8 - 9x^7 + 18x^6 + \dots$            | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0482 | abacDCBeDcgHC   | 3.402490... | $x^6 - 2x^5 - 3x^4 - 5x^3 + \dots$  | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0484 | abbchCBDCEDh    | 3.657501... | $x^{12} - 4x^{11} + x^{10} + x^9 - x^8 + 4x^7 - 6x^6 + \dots$               | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0487 | abcBBdceGh      | 2.344142... | $x^{14} - 2x^{13} - 2x^{12} + x^{11} + 3x^{10} + 4x^9 - 3x^8 + x^7 + \dots$ | $\{1^*, 3^9\}$      | Non-orientable |
| 12n0488 | abcdcEGh        | 1.842806... | $x^{12} - x^{11} - 2x^{10} - x^9 + 2x^8 + x^7 + 2x^6 + \dots$               | $\{1^*, 3^9\}$      | Non-orientable |
| 12n0491 | aBcBdCeDCgh     | 4.906479... | $x^{10} - 6x^9 + 5x^8 + 2x^7 - x^6 + \dots$                                 | $\{5^*, 3^5\}$      | Non-orientable |
| 12n0495 | abAbcdceghHCDE  | 3.284773... | $x^{12} - 4x^{11} + 3x^{10} - 3x^9 + 3x^8 - 2x^6 + \dots$                   | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0496 | abAbcdCEDgeHC   | 8.037866... | $x^4 - 10x^3 + 17x^2 + \dots$   | $\{6^*, 4^2\}$      | Orientable     |
| 12n0497 | abAbCdEF        | 5.475970... | $x^6 - 7x^5 + 9x^4 - 5x^3 + \dots$  | $\{8^*, 3^2\}$      | Non-orientable |
| 12n0499 | aBCBBEGhCBd     | 3.071432... | $x^{10} - 3x^9 - x^8 + 2x^7 - x^6 + 7x^5 + \dots$                           | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0505 | aBcdCCeGh       | 4.184847... | $x^8 - 6x^7 + 7x^6 + 5x^5 - 12x^4 + \dots$                                  | $\{6^*, 3^4\}$      | Non-orientable |
| 12n0506 | abcdCCEgh       | 4.235756... | $x^{12} - 6x^{11} + 6x^{10} + 9x^9 - 12x^8 + 6x^6 + \dots$                  | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0507 | abcbdCEDCgH     | 3.602650... | $x^{10} - 6x^9 + 10x^8 - 5x^7 - 3x^6 + 13x^5 + \dots$                       | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0510 | abcbdceHCBdCdEG | 8.051962... | $x^8 - 10x^7 + 15x^6 + 7x^5 - 13x^4 + \dots$                                | $\{6^*, 3^4\}$      | Non-orientable |
| 12n0514 | abcBDEGhcD      | 2.686374... | $x^{12} - x^{11} - 5x^{10} + x^9 + 4x^8 - 5x^7 - 9x^6 + \dots$              | $\{2^*, 3^8\}$      | Non-orientable |
| 12n0522 | abcDDEGh        | 2.006682... | $x^{10} - x^9 - 3x^8 + 3x^7 - x^6 - 2x^5 + \dots$                           | $\{1^*, 3^9\}$      | Non-orientable |
| 12n0525 | abCBeHcD        | 4.569495... | $x^8 - 7x^7 + 13x^6 - 9x^5 + 3x^4 + \dots$                                  | $\{7^*, 3^3\}$      | Non-orientable |
| 12n0528 | abAbcbdCeh      | 4.815994... | $x^6 - 2x^5 - 10x^4 - 15x^3 + \dots$  | $\{10^*\}$          | Orientable     |
| 12n0529 | abcBeHcD        | 6.560775... | $x^6 - 8x^5 + 10x^4 - 5x^3 + \dots$   | $\{8^*, 3^2\}$      | Non-orientable |
| 12n0536 | abAcBcDCegH     | 4.750268... | $x^{10} - 6x^9 + 6x^8 - x^7 + 6x^6 - 14x^5 + \dots$                         | $\{5^*, 3^5\}$      | Non-orientable |
| 12n0542 | abAccbDCeH      | 5.160059... | $x^6 - 7x^5 + 12x^4 - 15x^3 + \dots$  | $\{8^*, 3^2\}$      | Non-orientable |
| 12n0543 | abAcDGhCbCdE    | 4.204731... | $x^{12} - 4x^{11} - 3x^{10} + 9x^9 + 2x^8 - 9x^7 + 4x^6 + \dots$            | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0548 | abAcBGhCDDCed   | 4.874444... | $x^6 - 8x^5 + 20x^4 - 27x^3 + \dots$  | $\{6^*, 4^2\}$      | Orientable     |
| 12n0549 | abAcbdcbGGHCDE  | 2.618033... | $x^2 - 3x + \dots$  | $\{3^*, 3, 4^3\}$   | Non-orientable |
| 12n0552 | abCBegHCD       | 2.837597... | $x^8 - 4x^7 + 3x^6 + x^5 - x^4 + \dots$                                     | $\{3^*, 3^3, 4^2\}$ | Non-orientable |
| 12n0555 | abAbccDCEf      | 4.613470... | $x^3 - 5x^2 + 2x - 1$   | $\{8^*, 3^2\}$      | Non-orientable |
| 12n0558 | abcBcDEh        | 3.612971... | $x^{10} - 2x^9 - 4x^8 - 5x^7 - 5x^6 - x^5 + \dots$                          | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0577 | abcBdCeDCCgH    | 2.274681... | $x^{12} - x^{11} - 2x^{10} - x^9 - 3x^7 - 4x^6 + \dots$                     | $\{2^*, 3^8\}$      | Non-orientable |

Table C.3 – continued from previous page

|         |                |             |   |                     |                |
|---------|----------------|-------------|---|---------------------|----------------|
| 12n0579 | abAcbdcbGhCdE  | 2.297762... | $x^{12} - 2x^{11} - 3x^{10} + 7x^9 - 5x^8 - 2x^7 + 12x^6 + \dots$ | $\{2^*, 3^8\}$      | Non-orientable |
| 12n0584 | abcdHCdE       | 5.178144... | $x^8 - 6x^7 + 3x^6 + 7x^5 - 4x^4 + \dots$                         | $\{6^*, 3^4\}$      | Non-orientable |
| 12n0586 | abcDeHCD       | 4.892142... | $x^6 - 4x^5 - 2x^4 - 11x^3 + \dots$                               | $\{6^*, 3^4\}$      | Non-orientable |
| 12n0587 | abcDEHCd       | 6.618478... | $x^6 - 8x^5 + 10x^4 - 7x^3 + \dots$                               | $\{8^*, 3^2\}$      | Non-orientable |
| 12n0598 | aabAcBeHCd     | 4.825711... | $x^6 - 4x^5 - 2x^4 - 9x^3 + \dots$                                | $\{6^*, 3^4\}$      | Non-orientable |
| 12n0601 | abccEhCD       | 3.194009... | $x^{10} - 2x^9 - 2x^8 - 5x^7 - 3x^6 + 3x^5 + \dots$               | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0602 | abceHCCd       | 5.995485... | $x^4 - 6x^3 + x^2 + \dots$  | $\{6^*, 3^4\}$      | Non-orientable |
| 12n0612 | aBcdGhCDE      | 5.775688... | $x^6 - 9x^5 + 23x^4 - 29x^3 + \dots$                              | $\{6^*, 4^2\}$      | Orientable     |
| 12n0614 | abAcbdCBGhCdE  | 4.095491... | $x^{10} - 4x^9 - x^7 - 3x^6 + 3x^5 + \dots$                       | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0616 | aBBCBDCCDEgH   | 3.514937... | $x^{10} - 4x^9 + 2x^8 - x^7 - 3x^6 + 11x^5 + \dots$               | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0617 | abacbhCBdCeD   | 3.266437... | $x^8 - 4x^7 + 2x^6 + 2x^5 - 3x^4 + \dots$                         | $\{3^*, 3^3, 4^2\}$ | Non-orientable |
| 12n0621 | abCBegHcD      | 4.443697... | $x^{10} - 6x^9 + 8x^8 - 7x^7 + 14x^6 - 22x^5 + \dots$             | $\{5^*, 3^5\}$      | Non-orientable |
| 12n0628 | abCdCEDGhcBcH  | 5.444015... | $x^{10} - 7x^9 + 10x^8 - 9x^7 + 2x^6 + 9x^5 + \dots$              | $\{5^*, 3^5\}$      | Non-orientable |
| 12n0629 | aabAcbdCEh     | 2.806933... | $x^{10} - x^9 - 3x^8 - 4x^7 - 4x^6 - x^5 + \dots$                 | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0630 | abacGHCBDCEdC  | 3.057769... | $x^{12} - 3x^{11} - 2x^9 + 6x^8 - 5x^7 + 2x^6 + \dots$            | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0631 | abAcbgHCBDCeD  | 4.276772... | $x^{10} - 4x^9 - x^8 - x^7 + 4x^5 + \dots$                        | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0634 | abAcbdceDCCGh  | 3.956144... | $x^{12} - 4x^{11} - x^{10} + 4x^9 + 3x^8 - 2x^7 + x^6 + \dots$    | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0636 | abCBdcEGh      | 4.739517... | $x^{10} - 6x^9 + 6x^8 - x^7 + 6x^6 - 10x^5 + \dots$               | $\{5^*, 3^5\}$      | Non-orientable |
| 12n0660 | abacbdCehCdE   | 5.697466... | $x^4 - 4x^3 - 9x^2 + \dots$                                       | $\{10^*\}$          | Orientable     |
| 12n0672 | abAbcDEhCh     | 4.071338... | $x^{10} - 4x^9 - x^7 - 3x^6 + 10x^5 + \dots$                      | $\{4^*, 3^6\}$      | Non-orientable |
| 12n0690 | abcBGhcdEd     | 5.145185... | $x^{12} - 4x^{11} - 6x^{10} - x^9 + 6x^8 + 8x^7 + 10x^6 + \dots$  | $\{1^*, 3^9\}$      | Non-orientable |
| 12n0705 | abcBdcEDCGh    | 5.578686... | $x^8 - 7x^7 + 9x^6 - 10x^5 + 24x^4 + \dots$                       | $\{5^*, 3^5\}$      | Non-orientable |
| 12n0717 | abacDcBEDCGHcd | 3.358120... | $x^{12} - 3x^{11} - 3x^{10} + 7x^9 - 2x^8 - 6x^7 + 8x^6 + \dots$  | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0719 | abaccBDCEGHcdE | 3.449368... | $x^{12} - 4x^{11} + x^{10} + 4x^9 - 3x^8 - x^6 + \dots$           | $\{3^*, 3^7\}$      | Non-orientable |
| 12n0730 | abbcdbccdeGh   | 3.203980... | $x^{12} - 2x^{11} - 4x^{10} + 3x^8 - 2x^7 - 9x^6 + \dots$         | $\{2^*, 3^8\}$      | Non-orientable |
| 12n0732 | abcBDeHCdEg    | 5.200038... | $x^6 - 7x^5 + 10x^4 - 5x^3 + \dots$                               | $\{5^*, 3^5\}$      | Non-orientable |
| 12n0745 | abCdCBhcDE     | 6.379067... | $x^8 - 9x^7 + 21x^6 - 33x^5 + 41x^4 + \dots$                      | $\{7^*, 3^3\}$      | Non-orientable |
| 12n0749 | abcbdccceGh    | 1.864060... | $x^8 - x^7 - 2x^6 + 2x^4 + \dots$                                 | $\{1^*, 3^5, 4^2\}$ | Non-orientable |
| 12n0755 | abAcbdcbGHCD   | 6.579397... | $x^8 - 9x^7 + 16x^6 + 2x^5 - 17x^4 + \dots$                       | $\{6^*, 3^4\}$      | Non-orientable |
| 12n0757 | abacBDeDCHcd   | 6.112371... | $x^8 - 8x^7 + 12x^6 - 2x^5 - 5x^4 + \dots$                        | $\{7^*, 3^3\}$      | Non-orientable |

Table C.3 – continued from previous page

|         |                |             |   |                |                |
|---------|----------------|-------------|---|----------------|----------------|
| 12n0763 | abaCBAdCBehC   | 8.546721... | $x^6 - 10x^5 + 14x^4 - 15x^3 + \dots$                           | $\{8^*, 3^2\}$ | Non-orientable |
| 12n0768 | abbcBDEGh      | 2.242002... | $x^{10} - 3x^8 - 4x^7 - x^6 + x^5 + \dots$                      | $\{2^*, 3^8\}$ | Non-orientable |
| 12n0780 | abacbGhCBDCEd  | 5.750704... | $x^{10} - 7x^9 + 7x^8 + x^7 + x^6 - 4x^5 + \dots$               | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0798 | abAcBdEDCHCD   | 4.628918... | $x^8 - 5x^7 + x^6 + 4x^5 - 4x^4 + \dots$                        | $\{6^*, 3^4\}$ | Non-orientable |
| 12n0818 | abAcBdCCeH     | 6.338089... | $x^8 - 7x^7 + 3x^6 + 8x^5 - 4x^4 + \dots$                       | $\{6^*, 3^4\}$ | Non-orientable |
| 12n0826 | abcBdeHCdEG    | 5.871633... | $x^{10} - 9x^9 + 20x^8 - 7x^7 - 19x^6 + 26x^5 + \dots$          | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0835 | aabCBADCBEDCgH | 2.715718... | $x^{12} - 2x^{11} - 2x^{10} - x^9 + 2x^8 + 2x^7 + 2x^6 + \dots$ | $\{1^*, 3^9\}$ | Non-orientable |
| 12n0847 | abAcBdEDChcd   | 4.050167... | $x^6 - 4x^5 - 2x^4 + 8x^3 + \dots$                              | $\{6^*, 3^4\}$ | Non-orientable |
| 12n0870 | abAbCBDCEgH    | 3.367996... | $x^{12} - 4x^{11} + 2x^{10} + 2x^8 - 5x^7 + 12x^6 + \dots$      | $\{3^*, 3^7\}$ | Non-orientable |
| 12n0871 | abaCBAdcBEGhC  | 6.560288... | $x^{10} - 8x^9 + 10x^8 - 5x^7 + 12x^6 - 22x^5 + \dots$          | $\{5^*, 3^5\}$ | Non-orientable |
| 12n0878 | abacBdcBHCDc   | 7.584168... | $x^6 - 9x^5 + 12x^4 - 11x^3 + \dots$                            | $\{8^*, 3^2\}$ | Non-orientable |

Table C.4: Fibred knot complements with fibre  $S_{4,1}$ .

| Name              | Monodromy      | $\lambda(h)$ | $\mu_{\lambda(h)}(x)$  | Stratum             | Stable lamination |
|-------------------|----------------|--------------|--|---------------------|-------------------|
| 10 <sub>2</sub>   | abcdefgH       | 2.011287...  | $x^8 - 3x^7 + 3x^6 - 3x^5 + 3x^4 + \dots$  | $\{2^*, 8^2\}$      | Orientable        |
| 10 <sub>5</sub>   | abcdefGH       | 1.738874...  | $x^{10} - x^9 - 2x^7 + \dots$  | $\{2^*, 3^2, 7^2\}$ | Non-orientable    |
| 10 <sub>9</sub>   | abcdeFGH       | 1.635573...  | $x^6 - 2x^5 + 2x^4 - 3x^3 + \dots$   | $\{2^*, 4^2, 6^2\}$ | Orientable        |
| 10 <sub>17</sub>  | abcdEFGH       | 1.607506...  | $x^{10} - x^9 - 4x^5 + \dots$  | $\{2^*, 5^4\}$      | Non-orientable    |
| 10 <sub>46</sub>  | abcdefgJ       | 2.257161...  | $x^{16} - 3x^{15} + 3x^{14} - 4x^{13} + 4x^{12} - 5x^{11} + 4x^{10} - 4x^9 + 3x^8 + \dots$     | $\{2^*, 3^{12}\}$   | Non-orientable    |
| 10 <sub>47</sub>  | abcdeFGj       | 2.077144...  | $x^{16} - 3x^{15} + 4x^{14} - 7x^{13} + 9x^{12} - 11x^{11} + 12x^{10} - 13x^9 + 14x^8 + \dots$ | $\{2^*, 3^{12}\}$   | Non-orientable    |
| 10 <sub>48</sub>  | abcdEFGJ       | 1.860254...  | $x^{10} - 2x^9 + 2x^6 - 3x^5 + \dots$  | $\{2^*, 3^6, 5^2\}$ | Non-orientable    |
| 10 <sub>62</sub>  | abcdCedfEGHj   | 1.953754...  | $x^{14} - 2x^{13} + x^{12} - 3x^{11} + 3x^{10} - 2x^9 + 3x^8 - 4x^7 + \dots$                   | $\{2^*, 3^{12}\}$   | Non-orientable    |
| 10 <sub>64</sub>  | abcdCeDFEGHJ   | 1.781643...  | $x^6 - x^5 - x^4 + \dots$  | $\{2^*, 3^8, 4^2\}$ | Non-orientable    |
| 10 <sub>82</sub>  | abcdedFEGJ     | 2.172303...  | $x^{18} - 3x^{17} + 2x^{16} - x^{14} - x^{13} + 2x^{12} + 3x^{10} - 4x^9 + \dots$              | $\{3^*, 3^{11}\}$   | Non-orientable    |
| 10 <sub>85</sub>  | abcdeDfEgJ     | 2.338008...  | $x^{18} - 3x^{17} + 2x^{16} - 3x^{14} + x^{13} + 2x^{11} - 3x^{10} + 4x^9 + \dots$             | $\{3^*, 3^{11}\}$   | Non-orientable    |
| 10 <sub>91</sub>  | abcbDcDFEGJ    | 2.170965...  | $x^{18} - 3x^{17} + 6x^{15} - 3x^{14} - 5x^{13} + 10x^{11} + x^{10} - 15x^9 + \dots$           | $\{3^*, 3^{11}\}$   | Non-orientable    |
| 10 <sub>94</sub>  | abcBdCeDfEgJ   | 2.183459...  | $x^{18} - 3x^{17} + 6x^{15} - 3x^{14} - 7x^{13} + 6x^{12} + 6x^{11} - 3x^{10} - 5x^9 + \dots$  | $\{3^*, 3^{11}\}$   | Non-orientable    |
| 10 <sub>100</sub> | abcBdceDfGjeFG | 2.515824...  | $x^{18} - 3x^{17} + 5x^{15} - 4x^{14} - 5x^{13} + 8x^{12} - x^{11} - 5x^{10} + 6x^9 + \dots$   | $\{3^*, 3^{11}\}$   | Non-orientable    |

Table C.4 – continued from previous page

|                   |                |             |   |                     |                |
|-------------------|----------------|-------------|---|---------------------|----------------|
| 10 <sub>112</sub> | abcdCeDfGFEJef | 2.413138... | $x^{16} - 3x^{15} + 2x^{14} - x^{13} - x^{11} - 4x^{10} + 2x^9 - x^8 + \dots$                 | $\{4^*, 3^{10}\}$   | Non-orientable |
| 10 <sub>139</sub> | abcdCedfEghj   | 1.401268... | $x^6 - x^4 - x^3 + \dots$   | $\{14^*\}$          | Orientable     |
| 11a7              | abcdEFgJ       | 2.629562... | $x^{12} - 3x^{11} + 3x^9 - 2x^8 + 6x^6 + \dots$   | $\{4^*, 3^4, 5^2\}$ | Non-orientable |
| 11a9              | abcdefGJ       | 2.808151... | $x^{16} - 3x^{15} + x^{13} + x^{12} + x^{11} + x^9 + 2x^8 + \dots$                            | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a28             | abcBdCeDfEGJ   | 2.894665... | $x^{16} - 5x^{15} + 6x^{14} + 4x^{13} - 13x^{12} + 3x^{11} + 12x^{10} - 2x^9 - 13x^8 + \dots$ | $\{5^*, 3^9\}$      | Non-orientable |
| 11a33             | aBCDEfgJ       | 2.341758... | $x^{16} - 3x^{15} + x^{14} + 3x^{12} - 2x^{10} + 3x^9 + 2x^8 + \dots$                         | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a34             | abcDEfgJ       | 2.550965... | $x^{12} - 3x^{11} + 2x^{10} - 3x^9 + 4x^7 + 2x^6 + \dots$                                     | $\{4^*, 3^6, 4^2\}$ | Non-orientable |
| 11a35             | abCDEfgJ       | 2.338366... | $x^8 - x^7 - 2x^6 - x^5 - 3x^4 + \dots$   | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a40             | abcdEfgJ       | 2.678723... | $x^{10} - x^9 - x^8 - 6x^7 - 5x^6 - 8x^5 + \dots$   | $\{4^*, 3^4, 5^2\}$ | Non-orientable |
| 11a53             | abcdedFEgJ     | 3.194357... | $x^{16} - 5x^{15} + 8x^{14} - 8x^{13} + 3x^{12} - x^{11} + 5x^9 - 8x^8 + \dots$               | $\{5^*, 3^9\}$      | Non-orientable |
| 11a55             | abcdeFGJ       | 2.772695... | $x^8 - 3x^7 + 2x^6 - 5x^5 + 5x^4 + \dots$   | $\{4^*, 3^2, 6^2\}$ | Non-orientable |
| 11a62             | abcdeFgj       | 3.652080... | $x^{16} - 3x^{15} - 2x^{14} - x^{13} - x^{12} - x^{11} - 2x^{10} - x^9 + 2x^8 + \dots$        | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a66             | abcbdCeDfEGJ   | 2.916417... | $x^{16} - 5x^{15} + 6x^{14} + 4x^{13} - 13x^{12} + x^{11} + 16x^{10} - 19x^8 + \dots$         | $\{5^*, 3^9\}$      | Non-orientable |
| 11a68             | abcdeDfEGJ     | 3.086440... | $x^{16} - 5x^{15} + 8x^{14} - 8x^{13} + 5x^{12} - x^{11} + 3x^9 - 4x^8 + \dots$               | $\{5^*, 3^9\}$      | Non-orientable |
| 11a71             | abCdEDfegJ     | 2.713765... | $x^{12} - 2x^{11} - 2x^{10} - x^9 + 3x^8 + x^6 + \dots$                                       | $\{5^*, 3^9\}$      | Non-orientable |
| 11a72             | abCdEDGjeF     | 2.740266... | $x^{12} - 2x^{11} - 2x^{10} - x^9 + x^8 + 4x^7 - x^6 + \dots$                                 | $\{5^*, 3^9\}$      | Non-orientable |
| 11a73             | aBcdEDCfEdgFeJ | 2.899087... | $x^{14} - 5x^{13} + 8x^{12} - 7x^{11} + 6x^{10} - 6x^9 + x^8 + 5x^7 + \dots$                  | $\{6^*, 3^8\}$      | Non-orientable |
| 11a74             | aBCDEFGj       | 2.451540... | $x^{16} - 3x^{15} + 3x^{13} + x^{12} - x^{11} + x^9 + 2x^8 + \dots$                           | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a76             | abCBdcEDGjeF   | 2.775964... | $x^{16} - 5x^{15} + 6x^{14} + 4x^{13} - 11x^{12} - x^{11} + 14x^{10} + 2x^9 - 21x^8 + \dots$  | $\{5^*, 3^9\}$      | Non-orientable |
| 11a79             | abCBdcEDfegJ   | 2.825872... | $x^{16} - 5x^{15} + 6x^{14} + 4x^{13} - 11x^{12} - 3x^{11} + 16x^{10} + 6x^9 - 27x^8 + \dots$ | $\{5^*, 3^9\}$      | Non-orientable |
| 11a80             | aBCdEDfegJ     | 2.578276... | $x^{16} - 5x^{15} + 7x^{14} + x^{13} - 10x^{12} + 6x^{11} + 4x^9 - 9x^8 + \dots$              | $\{5^*, 3^9\}$      | Non-orientable |
| 11a81             | aBCdEDGjeF     | 2.609795... | $x^{16} - 5x^{15} + 7x^{14} + x^{13} - 12x^{12} + 12x^{11} - 12x^9 + 17x^8 + \dots$           | $\{5^*, 3^9\}$      | Non-orientable |
| 11a82             | abCDEFGj       | 2.474592... | $x^{12} - 3x^{11} + 2x^{10} - 3x^9 + 4x^8 - 2x^7 + \dots$                                     | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a83             | abCDefgj       | 2.704111... | $x^{12} - 3x^{11} + 3x^{10} - 8x^9 + 8x^8 - 9x^7 + 9x^6 + \dots$                              | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a86             | abcdCeDfEgHJ   | 2.589978... | $x^{16} - 3x^{15} + x^{14} - x^{13} + 3x^{12} - 2x^{10} + 4x^9 + 2x^8 + \dots$                | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a88             | abcdCedFEGHJ   | 2.538143... | $x^{10} - 2x^9 - 2x^8 + x^7 + x^6 + x^5 + \dots$  | $\{4^*, 3^6, 4^2\}$ | Non-orientable |
| 11a92             | aBCDefgj       | 2.545022... | $x^{10} - 2x^9 - 2x^8 + 2x^7 - 3x^5 + \dots$  | $\{4^*, 3^6, 4^2\}$ | Non-orientable |
| 11a99             | abCBdCeFEDgfeJ | 2.889341... | $x^{14} - 4x^{13} + x^{12} + 8x^{11} - 3x^{10} - 6x^9 + 2x^8 + 4x^7 + \dots$                  | $\{5^*, 3^9\}$      | Non-orientable |
| 11a106            | abcdCeDfEGHJ   | 2.542048... | $x^6 - x^5 - 2x^4 - 4x^3 + \dots$   | $\{4^*, 3^4, 5^2\}$ | Non-orientable |
| 11a108            | aBcDCedFEGHj   | 2.376274... | $x^{12} - 3x^{11} + 2x^{10} - 2x^9 + 2x^8 + x^7 - 4x^6 + \dots$                               | $\{4^*, 3^{10}\}$   | Non-orientable |

Table C.4 – continued from previous page

|        |                |             |  |                     |                |
|--------|----------------|-------------|--|---------------------|----------------|
| 11a109 | abcDCedFEGHj   | 2.457196... | $x^{12} - 3x^{11} + 3x^{10} - 7x^9 + 10x^8 - 11x^7 + 13x^6 + \dots$                          | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a125 | abAcbdCdedFEGJ | 3.113194... | $x^{14} - 3x^{13} - 2x^{11} + 3x^{10} - 3x^9 + 6x^8 + 3x^7 + \dots$                          | $\{5^*, 3^9\}$      | Non-orientable |
| 11a126 | abAcbdCdEFGJ   | 3.011164... | $x^{12} - 2x^{11} - 3x^{10} - 2x^8 + 5x^7 - 2x^6 + \dots$                                    | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a127 | abcBdcEDfegJ   | 3.092588... | $x^{14} - 3x^{13} - 3x^{10} + x^9 - 2x^8 + 5x^7 + \dots$                                     | $\{5^*, 3^9\}$      | Non-orientable |
| 11a129 | abcBdceDfEgJ   | 3.255868... | $x^{12} - 4x^{11} + x^{10} + 8x^9 - 11x^8 - 5x^7 + 19x^6 + \dots$                            | $\{5^*, 3^9\}$      | Non-orientable |
| 11a139 | abcdCedFEGHj   | 2.527324... | $x^8 - 3x^7 + 4x^6 - 8x^5 + 5x^4 + \dots$  | $\{4^*, 3^2, 4^4\}$ | Non-orientable |
| 11a142 | abcdCedfEghJ   | 3.593676... | $x^{12} - 3x^{11} - 3x^{10} + 3x^9 + x^8 - 2x^7 + \dots$                                     | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a147 | abcBdcEDGjeF   | 2.938946... | $x^{14} - 3x^{13} + x^{10} - x^9 + 6x^8 + 3x^7 + \dots$                                      | $\{5^*, 3^9\}$      | Non-orientable |
| 11a151 | abcBdceDGjEF   | 3.111500... | $x^{16} - 5x^{15} + 6x^{14} + 2x^{13} - 9x^{12} + 3x^{11} + 6x^{10} + 4x^9 - 15x^8 + \dots$  | $\{5^*, 3^9\}$      | Non-orientable |
| 11a158 | abAcBdCeDgjEEf | 3.087326... | $x^{16} - 3x^{15} - 2x^{14} + 5x^{13} + x^{12} + x^{11} - 2x^{10} - 5x^9 + 12x^8 + \dots$    | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a160 | abcBdCeFEDgfeJ | 2.913193... | $x^{14} - 3x^{13} + x^{10} + 3x^9 - 2x^8 + 9x^7 + \dots$                                     | $\{5^*, 3^9\}$      | Non-orientable |
| 11a163 | abcBdCeDfGjeFG | 3.038847... | $x^{16} - 5x^{15} + 6x^{14} + 2x^{13} - 9x^{12} + 9x^{11} - 2x^{10} - 10x^9 + 17x^8 + \dots$ | $\{5^*, 3^9\}$      | Non-orientable |
| 11a170 | abcBcdceFEDGJE | 3.279923... | $x^{14} - 3x^{13} - x^{12} + x^{11} - 4x^{10} + 6x^9 - 6x^8 + 13x^7 + \dots$                 | $\{5^*, 3^9\}$      | Non-orientable |
| 11a171 | abcdEDCfEdgFeJ | 3.142262... | $x^6 - 4x^5 + 4x^4 - 5x^3 + \dots$   | $\{6^*, 3^2, 4^3\}$ | Non-orientable |
| 11a174 | abcDEFGH       | 2.186019... | $x^8 - 3x^7 + 3x^6 - 5x^5 + 7x^4 + \dots$  | $\{4^*, 3^2, 6^2\}$ | Non-orientable |
| 11a175 | abcDEFGH       | 2.125727... | $x^8 - 2x^7 + x^6 - 3x^5 + 2x^4 + \dots$   | $\{4^*, 3^4, 5^2\}$ | Non-orientable |
| 11a176 | abcDEFgh       | 2.230352... | $x^8 - 3x^7 + 5x^6 - 9x^5 + 7x^4 + \dots$  | $\{4^*, 3^2, 4^4\}$ | Non-orientable |
| 11a177 | abcDEFgh       | 2.494798... | $x^{10} - x^9 - 6x^7 - 4x^6 - 8x^5 + \dots$  | $\{4^*, 3^4, 5^2\}$ | Non-orientable |
| 11a179 | aBCDEFGH       | 2.212584... | $x^8 - 3x^7 + x^6 + x^5 + x^4 + \dots$   | $\{4^*, 7^2\}$      | Non-orientable |
| 11a180 | abcDEFGH       | 2.256455... | $x^6 - 2x^5 - x^3 + \dots$   | $\{4^*, 4^2, 5^2\}$ | Non-orientable |
| 11a182 | abcdeFGH       | 2.710685... | $x^6 - 2x^5 - 5x^3 + \dots$  | $\{4^*, 3^2, 6^2\}$ | Non-orientable |
| 11a184 | abcdEFGH       | 2.408525... | $x^8 - 3x^7 + 3x^6 - 3x^5 - x^4 + \dots$   | $\{4^*, 4^2, 5^2\}$ | Non-orientable |
| 11a194 | aBcDCedfEghj   | 2.832720... | $x^{10} - 2x^9 - x^8 - 3x^7 - 2x^6 + \dots$  | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a203 | abcdeFgh       | 3.142262... | $x^6 - 4x^5 + 4x^4 - 5x^3 + \dots$   | $\{4^*, 3^2, 6^2\}$ | Non-orientable |
| 11a206 | aBcBDCEJeeFGIJ | 3.414909... | $x^8 - 3x^7 - x^6 - x^5 - x^4 + \dots$   | $\{4^*, 7^2\}$      | Non-orientable |
| 11a221 | abcdEFGj       | 2.761693... | $x^8 - 3x^7 + 2x^6 - 3x^5 - x^4 + \dots$   | $\{4^*, 4^2, 5^2\}$ | Non-orientable |
| 11a223 | abcDCedfEghj   | 3.346973... | $x^4 - 2x^3 - 4x^2 + \dots$  | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a232 | abcBcdeDFEDGHJ | 3.011970... | $x^{10} - 2x^9 - 3x^8 - x^7 + 8x^5 + \dots$  | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a248 | abcdCdedFEGJ   | 3.104624... | $x^{12} - 5x^{11} + 9x^{10} - 13x^9 + 12x^8 - 7x^7 + 7x^6 + \dots$                           | $\{5^*, 3^5, 4^2\}$ | Non-orientable |
| 11a255 | abcdCedfJEFG   | 3.285871... | $x^{16} - 5x^{15} + 6x^{14} - 5x^{12} + 3x^{11} + 8x^9 - 15x^8 + \dots$                      | $\{5^*, 3^9\}$      | Non-orientable |



Table C.4 – continued from previous page

|         |                |             |   |                     |                |
|---------|----------------|-------------|---|---------------------|----------------|
| 11a257  | abcDEFGj       | 2.689915... | $x^{12} - 3x^{11} + x^{10} - 2x^8 + x^7 + 3x^6 + \dots$   | $\{4^*, 3^6, 4^2\}$ | Non-orientable |
| 11a259  | abcDefgj       | 3.353251... | $x^{12} - 3x^{11} - x^{10} - 2x^8 - x^7 + 3x^6 + \dots$   | $\{4^*, 3^6, 4^2\}$ | Non-orientable |
| 11a261  | abcdEDfegJ     | 3.037400... | $x^{12} - 5x^{11} + 9x^{10} - 11x^9 + 4x^8 + 7x^7 - 11x^6 + \dots$                                  | $\{5^*, 3^5, 4^2\}$ | Non-orientable |
| 11a264  | abcdEDGjeF     | 2.960851... | $x^{12} - 5x^{11} + 9x^{10} - 11x^9 + 6x^8 + 3x^7 - 5x^6 + \dots$                                   | $\{5^*, 3^5, 4^2\}$ | Non-orientable |
| 11a269  | abcdCeDfgJeJEF | 3.354796... | $x^{16} - 5x^{15} + 5x^{14} + 5x^{13} - 14x^{12} + 10x^{11} + 4x^{10} - 14x^9 + 17x^8 + \dots$      | $\{5^*, 3^9\}$      | Non-orientable |
| 11a277  | abcbdCeDgJEF   | 3.111321... | $x^{16} - 5x^{15} + 6x^{14} + 2x^{13} - 9x^{12} + x^{11} + 14x^{10} - 19x^8 + \dots$                | $\{5^*, 3^9\}$      | Non-orientable |
| 11a282  | abcdCCeDfEGj   | 3.108211... | $x^{14} - 4x^{13} + 3x^{12} - x^{11} + x^{10} - x^9 + 2x^8 + \dots$                                 | $\{5^*, 3^9\}$      | Non-orientable |
| 11a289  | aabAcBDCeDFEGJ | 2.927123... | $x^{16} - 5x^{15} + 6x^{14} + 4x^{13} - 15x^{12} + 9x^{11} + 10x^{10} - 6x^9 - 9x^8 + \dots$        | $\{5^*, 3^9\}$      | Non-orientable |
| 11a293  | abcdefEEGj     | 3.464771... | $x^{16} - 3x^{15} - 2x^{14} + x^{13} + x^{12} + x^{11} - 2x^{10} + 3x^9 + 2x^8 + \dots$             | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a305  | abcBBdCedfEDGj | 2.992408... | $x^{14} - 4x^{13} + 2x^{12} + 5x^{11} - 7x^{10} + 2x^9 + 7x^8 - 11x^7 + \dots$                      | $\{5^*, 3^9\}$      | Non-orientable |
| 11a306  | abcDEfgh       | 2.442774... | $x^8 - 3x^7 + 5x^6 - 11x^5 + 9x^4 + \dots$  | $\{4^*, 3^2, 4^4\}$ | Non-orientable |
| 11a308  | abcdEfgh       | 3.061894... | $x^6 - 2x^5 - 2x^4 - 3x^3 + \dots$  | $\{4^*, 4^2, 5^2\}$ | Non-orientable |
| 11a316  | abcdCdEFGJ     | 3.205645... | $x^{12} - 3x^{11} - x^{10} + 2x^9 - 4x^8 + 3x^7 + x^6 + \dots$                                      | $\{4^*, 3^6, 4^2\}$ | Non-orientable |
| 11a326  | abcBdcEDfJEdFG | 3.411170... | $x^{16} - 5x^{15} + 4x^{14} + 8x^{13} - 11x^{12} - 3x^{11} + 14x^{10} + 2x^9 - 21x^8 + \dots$       | $\{5^*, 3^9\}$      | Non-orientable |
| 11a330  | abcdCedfEEGj   | 3.313281... | $x^{16} - 3x^{15} - 2x^{14} + 3x^{13} + x^{12} - x^{11} - 2x^{10} + 3x^9 + 2x^8 + \dots$            | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a346  | abcdCedfEgHGGj | 3.226641... | $x^{16} - 3x^{15} - 2x^{14} + 4x^{13} + x^{12} - 2x^{11} - 2x^{10} + 3x^9 + 2x^8 + \dots$           | $\{4^*, 3^{10}\}$   | Non-orientable |
| 11a348  | abcdcedJEDFEgf | 3.581722... | $x^{12} - 5x^{11} + 5x^{10} + 3x^9 - 10x^8 - 2x^7 + 12x^6 + \dots$                                  | $\{6^*, 3^8\}$      | Non-orientable |
| 11n13   | abcdefGj       | 2.242750... | $x^8 - 3x^7 + 2x^6 - x^5 + x^4 + \dots$   | $\{2^*, 4^6\}$      | Orientable     |
| 11n57   | aBCDEFGJ       | 2.060178... | $x^8 - 3x^7 + 2x^6 + x^5 - 3x^4 + \dots$  | $\{2^*, 6^3\}$      | Orientable     |
| 11n60   | abcDEFGJ       | 1.800171... | $x^8 - 3x^7 + 4x^6 - 5x^5 + 5x^4 + \dots$   | $\{2^*, 4^6\}$      | Orientable     |
| 11n61   | abcDEFGJ       | 1.826325... | $x^{10} - x^9 - x^8 - 2x^7 + x^6 + 2x^5 + \dots$  | $\{2^*, 3^6, 4^3\}$ | Non-orientable |
| 11n88   | abcdCedfEgHj   | 2.153721... | $x^4 - 3x^3 + 3x^2 + \dots$   | $\{2^*, 4^6\}$      | Orientable     |
| 11n104  | aBcDCeDFEGHJ   | 2.081018... | $x^4 - x^3 - 2x^2 + \dots$  | $\{2^*, 5^4\}$      | Non-orientable |
| 11n107  | abcDCeDFEGHJ   | 1.828870... | $x^{12} - x^{11} - 2x^9 - x^7 - 2x^6 + \dots$   | $\{2^*, 3^{12}\}$   | Non-orientable |
| 11n120  | aabAcBDCeDFEGJ | 2.084904... | $x^{18} - 3x^{17} + 2x^{16} - x^{14} + x^{13} + 3x^{10} - 3x^9 + \dots$                             | $\{3^*, 3^{11}\}$   | Non-orientable |
| 11n133  | abcBBdCedfEDgj | 2.153721... | $x^4 - 3x^3 + 3x^2 + \dots$   | $\{3^*, 3^5, 4^3\}$ | Non-orientable |
| 11n149  | abcdCCeDfEgJ   | 2.203299... | $x^{14} - x^{13} - 2x^{12} - 2x^{11} + x^{10} + x^8 + \dots$  | $\{3^*, 3^{11}\}$   | Non-orientable |
| 11n153  | abcbdCeDGJEF   | 2.032685... | $x^{18} - 3x^{17} + 2x^{16} - x^{15} + 2x^{14} + x^{13} - 2x^{12} - x^{11} + x^{10} - 2x^9 + \dots$ | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12a0016 | aBCDefGj       | 2.853035... | $x^{10} - 5x^9 + 7x^8 - 2x^7 - 5x^6 + 12x^5 + \dots$  | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0017 | abCDefGj       | 3.026085... | $x^{12} - 3x^{11} - 3x^9 + 4x^8 + 10x^7 + 6x^6 + \dots$   | $\{6^*, 3^8\}$      | Non-orientable |

Table C.4 – continued from previous page

|         |              |             |   |                     |                |
|---------|--------------|-------------|---|---------------------|----------------|
| 12a0018 | aBCDEFgj     | 2.896985... | $x^{12} - 3x^{11} - x^{10} + 2x^9 + 4x^8 + 3x^7 - x^6 + \dots$                      | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0019 | abcDEFgj     | 3.183205... | $x^{10} - 5x^9 + 7x^8 - 4x^7 - 3x^6 + 12x^5 + \dots$                                | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0020 | abcDEFgj     | 2.960717... | $x^{10} - 4x^9 + 2x^8 + 4x^7 - 2x^6 - x^5 + \dots$                                  | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0024 | abcDefGj     | 3.614185... | $x^{10} - 5x^9 + 5x^8 - 3x^6 + 12x^5 + \dots$                                       | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0026 | abcdeFgj     | 3.509705... | $x^8 - 3x^7 - 5x^5 - 3x^4 + \dots$  | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a0050 | abcdeDFeGJ   | 5.132791... | $x^8 - 4x^7 - 4x^6 - 8x^5 - 5x^4 + \dots$   | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a0069 | abCBdcEDgjeF | 3.384469... | $x^{14} - 7x^{13} + 16x^{12} - 10x^{11} - 17x^{10} + 25x^9 + 16x^8 - 47x^7 + \dots$ | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0070 | abCBdcEDgjeF | 3.862539... | $x^8 - 4x^7 + x^6 - 7x^4 + \dots$   | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0071 | abCBdceDfEGJ | 3.849547... | $x^{14} - 7x^{13} + 16x^{12} - 16x^{11} + x^{10} + 15x^9 - 10x^8 + x^7 + \dots$     | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0074 | abAcbdCdEFgJ | 3.640154... | $x^{12} - 3x^{11} - 2x^{10} - 3x^9 + 4x^8 + 6x^7 + 10x^6 + \dots$                   | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0077 | aBCdeFGj     | 2.947358... | $x^{12} - 3x^{11} - 3x^9 + 4x^8 + 12x^7 + 14x^6 + \dots$                            | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0079 | abAbCDEfgJ   | 2.923171... | $x^{14} - 5x^{13} + 6x^{12} + 3x^{11} - 7x^{10} - 3x^9 - 7x^8 + 20x^7 + \dots$      | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0084 | abCdeFGj     | 3.373735... | $x^{12} - 3x^{11} - 2x^{10} - x^9 + 6x^8 + 14x^7 + 14x^6 + \dots$                   | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0087 | aBcdeFGj     | 3.467982... | $x^{14} - 5x^{13} + 4x^{12} + 5x^{11} - x^{10} - x^9 - 5x^8 + 8x^7 + \dots$         | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0090 | abcBcDEfgJ   | 3.223035... | $x^{12} - 3x^{11} - 3x^9 + 2x^8 + 2x^6 + \dots$                                     | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0092 | abcBdceDFeGJ | 4.784290... | $x^8 - 4x^7 - x^6 - 12x^5 - 3x^4 + \dots$   | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0098 | aBCDeFGj     | 2.899617... | $x^{10} - 5x^9 + 9x^8 - 14x^7 + 21x^6 - 20x^5 + \dots$                              | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0099 | abCDeFGj     | 3.076441... | $x^{12} - 3x^{11} + 2x^{10} - 11x^9 + 12x^8 - 6x^7 + 26x^6 + \dots$                 | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0112 | abcDeFGj     | 3.660409... | $x^{10} - 5x^9 + 7x^8 - 12x^7 + 19x^6 - 16x^5 + \dots$                              | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0137 | aBCdEDfeGJ   | 3.311975... | $x^{14} - 7x^{13} + 17x^{12} - 17x^{11} + 16x^9 - 12x^8 + 5x^7 + \dots$             | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0138 | abcDEDfeGJ   | 3.868879... | $x^{10} - 7x^9 + 19x^8 - 38x^7 + 57x^6 - 63x^5 + \dots$                             | $\{7^*, 3^3, 4^2\}$ | Non-orientable |
| 12a0139 | abCdEDfeGJ   | 3.469146... | $x^{12} - 5x^{11} + 7x^{10} - 7x^9 + 4x^8 - x^7 + 3x^6 + \dots$                     | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0141 | aBCDEfgj     | 2.936062... | $x^8 - 3x^7 - x^5 + 5x^4 + \dots$   | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a0142 | abcDEfgj     | 3.412060... | $x^8 - 3x^7 + 2x^6 - 11x^5 + x^4 + \dots$   | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0149 | abCDEfgj     | 3.063872... | $x^8 - 3x^7 + 2x^6 - 7x^5 + 3x^4 + \dots$   | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0158 | abcdEfgj     | 4.365763... | $x^8 - 3x^7 - 4x^6 - 7x^5 - 5x^4 + \dots$   | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a0172 | aBCDEfGJ     | 3.668840... | $x^{14} - 5x^{13} + 4x^{12} + 3x^{11} + x^{10} - x^9 - x^8 + 12x^7 + \dots$         | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0173 | abcDEfGJ     | 3.820344... | $x^{10} - 5x^9 + 5x^8 - 2x^7 - 3x^6 + 14x^5 + \dots$                                | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0174 | abCDEfGJ     | 3.693392... | $x^{14} - 5x^{13} + 4x^{12} + 3x^{11} + x^{10} - 3x^9 - x^8 + 4x^7 + \dots$         | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0179 | abcdEfGJ     | 3.863029... | $x^{10} - 5x^9 + 5x^8 - 4x^7 + 7x^6 - 4x^5 + \dots$                                 | $\{6^*, 3^2, 5^2\}$ | Non-orientable |

Table C.4 – continued from previous page

|         |              |             |   |                     |                |
|---------|--------------|-------------|---|---------------------|----------------|
| 12a0184 | abCBdcEDfeGJ | 3.340556... | $x^{14} - 7x^{13} + 16x^{12} - 10x^{11} - 17x^{10} + 27x^9 + 16x^8 - 53x^7 + \dots$ | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0185 | abcBdcEDfeGJ | 3.745332... | $x^{12} - 5x^{11} + 6x^{10} - 6x^9 + 3x^8 + 4x^7 + x^6 + \dots$                     | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0186 | abcBdceDgjEF | 3.971725... | $x^{14} - 7x^{13} + 16x^{12} - 16x^{11} - 3x^{10} + 17x^9 - 4x^8 - 9x^7 + \dots$    | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0189 | aBCdEDgjeF   | 3.250713... | $x^{14} - 7x^{13} + 17x^{12} - 17x^{11} + 2x^{10} + 12x^9 - 18x^8 + 19x^7 + \dots$  | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0190 | abcdEDgjeF   | 3.922068... | $x^{10} - 7x^9 + 19x^8 - 38x^7 + 55x^6 - 61x^5 + \dots$                             | $\{7^*, 3^3, 4^2\}$ | Non-orientable |
| 12a0191 | abCdEDgjeF   | 3.435140... | $x^{12} - 5x^{11} + 7x^{10} - 7x^9 + 6x^8 - 5x^7 + 5x^6 + \dots$                    | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0193 | abcdeFgJ     | 3.881772... | $x^8 - 7x^7 + 16x^6 - 19x^5 + 19x^4 + \dots$  | $\{6^*, 6^2\}$      | Orientable     |
| 12a0200 | abcdCdeFgJ   | 3.805930... | $x^{10} - 5x^9 + 5x^8 - 2x^7 - x^6 + 8x^5 + \dots$                                  | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0202 | aBccDCedFEgJ | 3.385413... | $x^{14} - 7x^{13} + 17x^{12} - 17x^{11} - 2x^{10} + 26x^9 - 42x^8 + 47x^7 + \dots$  | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0203 | aBCdefgJ     | 3.003487... | $x^{14} - 5x^{13} + 6x^{12} - x^{11} + 3x^{10} + x^9 - 3x^8 + \dots$                | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0207 | abAbCDEFGj   | 2.996510... | $x^{12} - 7x^{11} + 18x^{10} - 21x^9 + 10x^8 - 4x^7 + 4x^6 + \dots$                 | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0209 | aBCDCedFGjEf | 3.436096... | $x^{14} - 7x^{13} + 16x^{12} - 12x^{11} - 9x^{10} + 19x^9 + 18x^8 - 53x^7 + \dots$  | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0213 | aBCdedFEGJ   | 3.361573... | $x^{12} - 5x^{11} + 7x^{10} - 7x^9 + 4x^8 + 9x^7 - 3x^6 + \dots$                    | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0214 | aBCdeDfEgJ   | 3.299793... | $x^{12} - 5x^{11} + 7x^{10} - 7x^9 + 6x^8 + 5x^7 + 3x^6 + \dots$                    | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0215 | abcdCedFEgHj | 3.057688... | $x^8 - 4x^7 + 2x^6 + 3x^5 - 2x^4 + \dots$   | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0216 | abcdCedfEgHJ | 3.780912... | $x^{14} - 5x^{13} + 4x^{12} + 2x^{11} + x^{10} - x^8 + 12x^7 + \dots$               | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0217 | aBcDCedFEghJ | 2.831405... | $x^{12} - 3x^{11} - x^{10} + 3x^9 + 4x^8 + x^7 - 9x^6 + \dots$                      | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0219 | abcdCedFEghJ | 3.345797... | $x^8 - 3x^7 + 2x^6 - 10x^5 + x^4 + \dots$   | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0220 | aBcDCedfEgHj | 3.065132... | $x^6 - 4x^5 + 3x^4 - x^3 + \dots$   | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0222 | aBcDCeDFgjEf | 3.486971... | $x^{14} - 7x^{13} + 16x^{12} - 14x^{11} + 3x^{10} - 3x^9 + 22x^8 - 37x^7 + \dots$   | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0224 | abCBdCEDGjeF | 3.390248... | $x^{14} - 7x^{13} + 16x^{12} - 12x^{11} - 9x^{10} + 27x^9 - 18x^8 + 5x^7 + \dots$   | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0225 | abcDCedFEghJ | 3.007511... | $x^{10} - 4x^9 + 3x^8 - x^7 + 2x^6 + 2x^5 + \dots$                                  | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0228 | aBcDCeDfgj   | 3.546734... | $x^{14} - 5x^{13} + 4x^{12} + 5x^{11} - 5x^{10} + 9x^9 - 11x^8 + 8x^7 + \dots$      | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0233 | abCBdCEDfegJ | 3.289176... | $x^{14} - 7x^{13} + 16x^{12} - 12x^{11} - 5x^{10} + 9x^9 + 18x^8 - 41x^7 + \dots$   | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0242 | abCDefgJ     | 3.118827... | $x^{10} - 5x^9 + 9x^8 - 16x^7 + 25x^6 - 24x^5 + \dots$                              | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0245 | abCDeeFEgJ   | 3.614286... | $x^{12} - 4x^{11} - x^{10} + 8x^9 + 4x^8 - x^7 - 18x^6 + \dots$                     | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0246 | aBCdEFGJ     | 3.674658... | $x^{14} - 5x^{13} + 4x^{12} + 3x^{11} + x^{10} - x^9 + x^8 - 4x^7 + \dots$          | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0250 | abcdCedfEgHj | 3.583349... | $x^{12} - 4x^{11} - x^{10} + 7x^9 + 9x^8 - 4x^7 - 12x^6 + \dots$                    | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0258 | aBCDefgJ     | 2.949787... | $x^4 - 4x^3 + 4x^2 - 3x + 1$  | $\{6^*, 4^4\}$      | Orientable     |
| 12a0260 | aBCDeeFEgJ   | 3.532957... | $x^{12} - 3x^{11} - 3x^{10} + 2x^9 + 6x^8 + 3x^7 - x^6 + \dots$                     | $\{6^*, 3^8\}$      | Non-orientable |

Table C.4 – continued from previous page

|         |                |             |  |                     |                |
|---------|----------------|-------------|--|---------------------|----------------|
| 12a0261 | abCBdceFEDgfEJ | 3.324478... | $x^{14} - 7x^{13} + 16x^{12} - 12x^{11} - 5x^{10} + 15x^9 - 22x^8 + 27x^7 + \dots$ | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0262 | aBcDCeedGJEF   | 3.466330... | $x^{14} - 5x^{13} + 3x^{12} + 12x^{11} - 12x^{10} - 9x^9 + 10x^8 - 2x^7 + \dots$   | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0264 | aBcdEDCjeDfEGF | 3.787264... | $x^{12} - 7x^{11} + 16x^{10} - 16x^9 + 3x^8 + 13x^7 - 15x^6 + \dots$               | $\{8^*, 3^6\}$      | Non-orientable |
| 12a0265 | aBcDCeDfEDgj   | 3.434469... | $x^{12} - 5x^{11} + 7x^{10} - 9x^9 + 12x^8 - 5x^7 + 15x^6 + \dots$                 | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0271 | abcBcDEFGj     | 3.308005... | $x^{14} - 5x^{13} + 6x^{12} - x^{11} + x^{10} - 7x^9 - 3x^8 + 12x^7 + \dots$       | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0278 | abcdCedfEGHJ   | 3.777951... | $x^8 - 4x^7 + x^5 + 8x^4 + \dots$  | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a0280 | aBcDCedFEghj   | 2.965572... | $x^4 - 3x^3 + x^2 + \dots$   | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0281 | abcDCedFEghj   | 3.254263... | $x^4 - 3x^3 + \dots$   | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0282 | aBcdedFEGJ     | 3.625293... | $x^{12} - 6x^{11} + 8x^{10} + 5x^9 - 9x^8 - 8x^7 + 17x^6 + \dots$                  | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0283 | aBcdeDfEgJ     | 3.614341... | $x^{12} - 6x^{11} + 8x^{10} + 5x^9 - 11x^8 + 4x^7 - x^6 + \dots$                   | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0288 | abAbCdEDfegJ   | 3.474468... | $x^{14} - 7x^{13} + 17x^{12} - 17x^{11} + 2x^9 + 22x^8 - 35x^7 + \dots$            | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0298 | aBcdCedfEGHj   | 3.553375... | $x^{12} - 7x^{11} + 17x^{10} - 23x^9 + 30x^8 - 39x^7 + 43x^6 + \dots$              | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0299 | abCdefgJ       | 3.407140... | $x^{14} - 5x^{13} + 4x^{12} + 5x^{11} + x^{10} - 5x^9 - x^8 + 4x^7 + \dots$        | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0304 | aBcdefgJ       | 3.475722... | $x^{12} - 7x^{11} + 16x^{10} - 15x^9 + 8x^8 - 6x^7 + 8x^6 + \dots$                 | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0305 | aBcdEFGJ       | 3.713544... | $x^{12} - 7x^{11} + 17x^{10} - 24x^9 + 30x^8 - 33x^7 + 33x^6 + \dots$              | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0316 | abAcBdCdcdFEGJ | 3.678038... | $x^{12} - 5x^{11} + 6x^{10} - 6x^9 + 7x^8 - 4x^7 + 11x^6 + \dots$                  | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0318 | abAcBdCdEFGJ   | 4.108992... | $x^{14} - 5x^{13} + 2x^{12} + 7x^{11} - x^{10} + 3x^9 - 9x^8 + 8x^7 + \dots$       | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0328 | abcBdCEDfegJ   | 3.494453... | $x^{12} - 5x^{11} + 6x^{10} - 4x^9 + 3x^8 + 6x^7 + x^6 + \dots$                    | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0331 | abAbCDefgj     | 3.687363... | $x^{10} - 6x^9 + 12x^8 - 16x^7 + 14x^6 - 11x^5 + \dots$                            | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0333 | abcBdCEDGjeF   | 3.512400... | $x^{12} - 5x^{11} + 6x^{10} - 4x^9 + 3x^8 + 4x^7 + 3x^6 + \dots$                   | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0334 | abAbCdEDGjeF   | 3.447022... | $x^{14} - 7x^{13} + 17x^{12} - 17x^{11} - 2x^{10} + 20x^9 - 26x^8 + 27x^7 + \dots$ | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0351 | abcBcdcdFEDGHJ | 3.684544... | $x^{12} - 4x^{11} - x^{10} + 7x^9 + 5x^8 - 20x^6 + \dots$                          | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0358 | abcDCeDfEDgj   | 3.929029... | $x^{12} - 5x^{11} + 5x^{10} - 5x^9 + 6x^8 + 3x^7 + 9x^6 + \dots$                   | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0359 | abcdEDCjeDfEGF | 4.086436... | $x^{10} - 6x^9 + 10x^8 - 13x^7 + 21x^6 - 22x^5 + \dots$                            | $\{8^*, 3^6\}$      | Non-orientable |
| 12a0373 | abcdCedFEghj   | 4.162434... | $x^6 - 4x^5 + x^4 - 7x^3 + \dots$  | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0374 | aBcDCedfEGHj   | 3.078856... | $x^{10} - 5x^9 + 9x^8 - 15x^7 + 21x^6 - 18x^5 + \dots$                             | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0377 | abCdedFEGJ     | 3.836860... | $x^{12} - 5x^{11} + 5x^{10} - 3x^9 + 2x^8 + 5x^7 + 3x^6 + \dots$                   | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0382 | aBcDCeDFEgHJ   | 3.640630... | $x^8 - 6x^7 + 9x^6 - 6x^4 + \dots$   | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0383 | abcDCeDFEgHJ   | 3.757352... | $x^{14} - 5x^{13} + 4x^{12} + 2x^{11} + 3x^{10} - 4x^9 - x^8 + 4x^7 + \dots$       | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0396 | abAbcDCedFEgHj | 3.233701... | $x^{12} - 7x^{11} + 19x^{10} - 31x^9 + 42x^8 - 59x^7 + 69x^6 + \dots$              | $\{6^*, 3^8\}$      | Non-orientable |

Table C.4 – continued from previous page

|         |                |             |  |                     |                |
|---------|----------------|-------------|--|---------------------|----------------|
| 12a0398 | abcDCeDfgj     | 4.082983... | $x^{14} - 5x^{13} + 2x^{12} + 7x^{11} + x^{10} - x^9 - 5x^8 + 4x^7 + \dots$        | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0402 | abcDCeDFgjEf   | 3.698403... | $x^{12} - 5x^{11} + 6x^{10} - 6x^9 + 5x^8 + 2x^7 + 5x^6 + \dots$                   | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0413 | abcDCedFGjEf   | 3.846807... | $x^{12} - 5x^{11} + 6x^{10} - 8x^9 + 7x^8 + 9x^6 + \dots$                          | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0415 | abAcBdCedfJEFG | 4.069438... | $x^{14} - 7x^{13} + 14x^{12} - 8x^{11} - 3x^{10} + 3x^9 + 14x^8 - 29x^7 + \dots$   | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0416 | abCBdCeDGjEF   | 3.997660... | $x^{14} - 7x^{13} + 14x^{12} - 6x^{11} - 13x^{10} + 23x^9 - 10x^8 - 3x^7 + \dots$  | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0418 | abAcBdCeDfEDGj | 3.866874... | $x^{14} - 7x^{13} + 16x^{12} - 16x^{11} - x^{10} + 25x^9 - 30x^8 + 25x^7 + \dots$  | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0436 | abCdeFGJ       | 4.214086... | $x^{14} - 5x^{13} + 2x^{12} + 5x^{11} + 3x^{10} - 3x^9 - x^8 + \dots$              | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0445 | abcDCedfEGHj   | 3.599298... | $x^{10} - 5x^9 + 7x^8 - 11x^7 + 17x^6 - 14x^5 + \dots$                             | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0446 | abCdeDfEgJ     | 3.780226... | $x^{12} - 5x^{11} + 5x^{10} - 3x^9 + 4x^8 + 5x^7 + 5x^6 + \dots$                   | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0453 | abcdCeDFeGHJ   | 3.776723... | $x^8 - 7x^7 + 18x^6 - 30x^5 + 37x^4 + \dots$                                       | $\{6^*, 4^4\}$      | Orientable     |
| 12a0456 | abCBdCeDfEgJ   | 3.928950... | $x^{14} - 7x^{13} + 14x^{12} - 6x^{11} - 9x^{10} + 9x^9 + 14x^8 - 33x^7 + \dots$   | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0457 | abCBdCEFGjeDfe | 3.882349... | $x^{14} - 7x^{13} + 14x^{12} - 4x^{11} - 21x^{10} + 37x^9 - 26x^8 + 13x^7 + \dots$ | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0464 | abAcBdCeeDFEGJ | 3.611123... | $x^{10} - 4x^9 + 6x^7 - 4x^6 + 3x^5 + \dots$                                       | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0468 | abcBdCeDfjEFgF | 4.797492... | $x^{12} - 4x^{11} - 4x^{10} + 3x^8 + 4x^7 + 3x^6 + \dots$                          | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0469 | abcdCeedGJEF   | 3.740393... | $x^{12} - 3x^{11} - 3x^{10} + 2x^8 + 5x^7 - 3x^6 + \dots$                          | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0470 | abcBdceFEDgfeJ | 3.582809... | $x^{12} - 5x^{11} + 6x^{10} - 4x^9 + 3x^8 - 2x^7 + x^6 + \dots$                    | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0473 | aBcdEDCGjEdjeF | 3.833187... | $x^{14} - 7x^{13} + 15x^{12} - 9x^{11} - 14x^{10} + 32x^9 - 34x^8 + 31x^7 + \dots$ | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0476 | abAcBdCeDgjEEf | 4.129362... | $x^{14} - 5x^{13} + x^{12} + 12x^{11} - 4x^{10} - 7x^9 + 6x^8 + 2x^7 + \dots$      | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0478 | abCBdceDfGjeFG | 3.890211... | $x^{10} - 4x^9 - x^8 + 6x^7 - x^6 - 3x^5 + \dots$                                  | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0479 | abbcdCBeDcFEgJ | 3.566571... | $x^{14} - 7x^{13} + 17x^{12} - 19x^{11} + 4x^{10} + 20x^9 - 40x^8 + 47x^7 + \dots$ | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0480 | abccDCedFEgJ   | 3.730283... | $x^{12} - 5x^{11} + 7x^{10} - 11x^9 + 12x^8 - 13x^7 + 17x^6 + \dots$               | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0485 | abCBdCedfEDCgj | 4.236774... | $x^{14} - 7x^{13} + 13x^{12} - 3x^{11} - 16x^{10} + 28x^9 - 22x^8 + 13x^7 + \dots$ | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0486 | aBcdCeDfGFEJef | 3.996778... | $x^6 - 8x^5 + 22x^4 - 29x^3 + \dots$   | $\{8^*, 4^3\}$      | Orientable     |
| 12a0487 | abAbcbDCedFEgJ | 3.702743... | $x^{14} - 7x^{13} + 16x^{12} - 14x^{11} - 5x^{10} + 17x^9 + 14x^8 - 43x^7 + \dots$ | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0488 | abcdCeDfEEGj   | 3.770112... | $x^{12} - 3x^{11} - 3x^{10} + 2x^8 - x^7 - 5x^6 + \dots$                           | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0493 | abcdCeDfEgHGGj | 3.865506... | $x^{12} - 3x^{11} - 3x^{10} - x^9 - 3x^7 - 7x^6 + \dots$                           | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0497 | abCDEfgh       | 2.823507... | $x^{10} - 5x^9 + 10x^8 - 18x^7 + 26x^6 - 24x^5 + \dots$                            | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0498 | abCDefgh       | 2.733999... | $x^{10} - 5x^9 + 10x^8 - 18x^7 + 28x^6 - 28x^5 + \dots$                            | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0499 | abCDefGH       | 2.882619... | $x^{12} - 3x^{11} + 3x^{10} - 13x^9 + 16x^8 - 12x^7 + 32x^6 + \dots$               | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0500 | aBCdefgh       | 2.774031... | $x^8 - 4x^7 + 3x^6 - x^5 + 6x^4 + \dots$   | $\{6^*, 3^2, 5^2\}$ | Non-orientable |

Table C.4 – continued from previous page

|         |                |             |   |                     |                |
|---------|----------------|-------------|---|---------------------|----------------|
| 12a0501 | abcDEfgh       | 2.982557... | $x^{10} - 5x^9 + 10x^8 - 20x^7 + 32x^6 - 32x^5 + \dots$ | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0503 | abAbCDEFGh     | 2.618033... | $x^2 - 3x + \dots$                                      | $\{6^*, 6^2\}$      | Orientable     |
| 12a0505 | abAbCDEfgh     | 2.990303... | $x^6 - 6x^5 + 14x^4 - 19x^3 + \dots$                    | $\{6^*, 4^4\}$      | Orientable     |
| 12a0506 | aBCDefgH       | 2.618033... | $x^2 - 3x + \dots$                                      | $\{6^*, 4^4\}$      | Orientable     |
| 12a0512 | abCDEfGH       | 3.438994... | $x^{10} - 5x^9 + 6x^8 - 4x^7 + 8x^6 - 8x^5 + \dots$     | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a0515 | abAbCDEFGh     | 2.719868... | $x^{10} - 5x^9 + 8x^8 - 6x^7 + 8x^6 - 16x^5 + \dots$    | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a0516 | abcBcDEFGh     | 2.896817... | $x^{10} - 5x^9 + 10x^8 - 16x^7 + 20x^6 - 24x^5 + \dots$ | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0517 | abCdefgH       | 3.191095... | $x^{10} - 5x^9 + 6x^8 - 2x^7 + 6x^6 - 8x^5 + \dots$     | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a0521 | aBcdefgH       | 3.423869... | $x^8 - 7x^7 + 17x^6 - 21x^5 + 21x^4 + \dots$            | $\{6^*, 6^2\}$      | Orientable     |
| 12a0528 | abCdefGH       | 3.254263... | $x^4 - 3x^3 + \dots$                                    | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0535 | abcDefGH       | 3.411514... | $x^{10} - 5x^9 + 8x^8 - 14x^7 + 22x^6 - 20x^5 + \dots$  | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0536 | abcdEfgh       | 3.481370... | $x^8 - 4x^7 + x^6 + x^5 + 6x^4 + \dots$                 | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a0541 | abcDefgh       | 3.221693... | $x^6 - 6x^5 + 12x^4 - 13x^3 + \dots$                    | $\{6^*, 4^4\}$      | Orientable     |
| 12a0561 | abAcbdCdEFgh   | 3.254263... | $x^4 - 3x^3 + \dots$                                    | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0565 | abcBcDEfgh     | 3.380550... | $x^{10} - 5x^9 + 10x^8 - 22x^7 + 30x^6 - 32x^5 + \dots$ | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0569 | abAbCDefgh     | 3.531622... | $x^{10} - 5x^9 + 8x^8 - 12x^7 + 8x^6 - 4x^5 + \dots$    | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a0579 | abcDEfGH       | 3.449276... | $x^{10} - 5x^9 + 8x^8 - 14x^7 + 20x^6 - 16x^5 + \dots$  | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0583 | abcDEFGH       | 3.490423... | $x^8 - 7x^7 + 19x^6 - 33x^5 + 41x^4 + \dots$            | $\{6^*, 4^4\}$      | Orientable     |
| 12a0584 | abcdEFgH       | 3.656233... | $x^{10} - 5x^9 + 6x^8 - 6x^7 + 8x^6 - 4x^5 + \dots$     | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a0621 | abcdeDFEGj     | 4.716597... | $x^8 - 4x^7 - 2x^6 - 6x^5 - x^4 + \dots$                | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a0629 | aabAcBDCEdGJEf | 3.624685... | $x^6 - 3x^5 - 8x^3 + \dots$                             | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0630 | abcbDCeDfEgJ   | 3.971716... | $x^8 - 4x^7 + 3x^6 - 12x^5 + 5x^4 + \dots$              | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0637 | abcdCCedFEGj   | 4.201074... | $x^8 - 4x^7 + 2x^6 - 12x^5 + 3x^4 + \dots$              | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0649 | abcdEfGH       | 4.054767... | $x^{10} - 5x^9 + 4x^8 - 2x^7 + 6x^6 - 4x^5 + \dots$     | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a0651 | abcdeFgH       | 4.214851... | $x^8 - 7x^7 + 15x^6 - 17x^5 + 17x^4 + \dots$            | $\{6^*, 6^2\}$      | Orientable     |
| 12a0662 | abcBdCeDFEGj   | 3.998317... | $x^8 - 4x^7 + 3x^6 - 12x^5 + 3x^4 + \dots$              | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0664 | abcdeDgJEf     | 4.620239... | $x^8 - 4x^7 - 2x^6 - 4x^5 + x^4 + \dots$                | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a0681 | abcDefgJ       | 3.684918... | $x^8 - 7x^7 + 18x^6 - 29x^5 + 35x^4 + \dots$            | $\{6^*, 4^4\}$      | Orientable     |
| 12a0683 | aBcdCeDFEGHJ   | 4.324188... | $x^{10} - 6x^9 + 7x^8 + x^7 + 2x^6 - 8x^5 + \dots$      | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0702 | abCBdCefegFhJE | 4.072445... | $x^{10} - 6x^9 + 7x^8 + 5x^7 - 6x^6 + \dots$            | $\{6^*, 3^8\}$      | Non-orientable |

Table C.4 – continued from previous page

|         |                |             |   |                     |                |
|---------|----------------|-------------|---|---------------------|----------------|
| 12a0706 | abcDCeDffGFEhJ | 3.804388... | $x^{14} - 5x^{13} + 4x^{12} + 2x^{11} + x^{10} - 13x^8 + 16x^7 + \dots$         | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0708 | aBcDCeDffGFEhJ | 3.444353... | $x^{12} - 7x^{11} + 16x^{10} - 12x^9 - 8x^8 + 23x^7 - 28x^6 + \dots$            | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0768 | abcdCeDFgjEF   | 4.870988... | $x^{12} - 4x^{11} - 4x^{10} - 2x^9 + 3x^8 + 4x^7 + 3x^6 + \dots$                | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0776 | abcBdceDGJeF   | 4.493926... | $x^8 - 4x^7 + x^6 - 14x^5 + x^4 + \dots$  | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0777 | abAcbdCedfEgFJ | 4.781542... | $x^{12} - 4x^{11} - 4x^{10} + 5x^8 + 4x^7 + 3x^6 + \dots$                       | $\{6^*, 3^8\}$      | Non-orientable |
| 12a0886 | abcDeeFEgJ     | 3.972374... | $x^{10} - 5x^9 + 5x^8 - 4x^7 - x^6 + 10x^5 + \dots$                             | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a0901 | abcdCedfGFEJeF | 5.363170... | $x^{14} - 8x^{13} + 17x^{12} - 16x^{11} + x^{10} + 16x^9 - 11x^8 + x^7 + \dots$ | $\{7^*, 3^7\}$      | Non-orientable |
| 12a0918 | abcdCedfGFEJEF | 5.070550... | $x^{10} - 8x^9 + 21x^8 - 41x^7 + 61x^6 - 67x^5 + \dots$                         | $\{7^*, 3^3, 4^2\}$ | Non-orientable |
| 12a0968 | abAcBdCeDfEfGJ | 4.857213... | $x^{10} - 6x^9 + 6x^8 - 2x^7 - 4x^6 + 16x^5 + \dots$                            | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a1039 | abcDeFGH       | 4.025599... | $x^8 - 7x^7 + 17x^6 - 27x^5 + 33x^4 + \dots$                                    | $\{6^*, 4^4\}$      | Orientable     |
| 12a1045 | abcdeFgJef     | 4.732056... | $x^6 - 3x^5 - 6x^4 - 9x^3 + \dots$  | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a1049 | abcdCedjEDFEgF | 5.272225... | $x^{12} - 7x^{11} + 9x^{10} + 2x^9 - 8x^8 + 2x^7 + 3x^6 + \dots$                | $\{7^*, 3^7\}$      | Non-orientable |
| 12a1070 | abcBDCeDfEgj   | 4.000331... | $x^{12} - 4x^{11} + x^{10} - 6x^9 + 4x^8 + 14x^7 + 4x^6 + \dots$                | $\{6^*, 3^8\}$      | Non-orientable |
| 12a1074 | abcDCCedfEgj   | 4.490493... | $x^{12} - 4x^{11} - 3x^{10} + 2x^9 + 6x^8 + 4x^7 + 3x^6 + \dots$                | $\{6^*, 3^8\}$      | Non-orientable |
| 12a1076 | abcBDCedFEGj   | 3.927254... | $x^{12} - 4x^{11} + x^{10} - 4x^9 + 2x^8 + 8x^7 + 8x^6 + \dots$                 | $\{6^*, 3^8\}$      | Non-orientable |
| 12a1080 | abcDeDfEgj     | 4.451075... | $x^{10} - 6x^9 + 7x^8 - 5x^6 + 14x^5 + \dots$                                   | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a1081 | abcbDCeDFEGj   | 4.421450... | $x^{10} - 6x^9 + 8x^8 - 4x^7 - 6x^6 + 18x^5 + \dots$                            | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a1082 | abcdCedfGJef   | 4.747595... | $x^{10} - 6x^9 + 6x^8 - 4x^6 + 14x^5 + \dots$                                   | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a1084 | abcdCCedfEgJ   | 5.044678... | $x^{14} - 6x^{13} + 4x^{12} + 4x^{11} + x^{10} - 2x^9 - x^8 + 14x^7 + \dots$    | $\{6^*, 3^8\}$      | Non-orientable |
| 12a1087 | abcBdCedFEGJ   | 4.278545... | $x^{10} - 6x^9 + 8x^8 - 2x^7 - 6x^6 + 14x^5 + \dots$                            | $\{6^*, 3^4, 4^2\}$ | Non-orientable |
| 12a1093 | aabAcBDCedGJEF | 4.426385... | $x^{12} - 4x^{11} - 2x^{10} + x^8 + 4x^7 + 5x^6 + \dots$                        | $\{6^*, 3^8\}$      | Non-orientable |
| 12a1096 | abcBDCedgJef   | 3.878162... | $x^{12} - 4x^{11} + x^{10} - 4x^9 + 4x^8 + 12x^7 + 4x^6 + \dots$                | $\{6^*, 3^8\}$      | Non-orientable |
| 12a1104 | aBcBdCeDgjEf   | 4.330206... | $x^{10} - 3x^9 - 6x^8 - x^7 + 6x^6 + 11x^5 + \dots$                             | $\{6^*, 3^8\}$      | Non-orientable |
| 12a1141 | abcdeDGJeF     | 4.819864... | $x^8 - 4x^7 - 2x^6 - 8x^5 - 5x^4 + \dots$                                       | $\{6^*, 3^2, 5^2\}$ | Non-orientable |
| 12a1153 | abcdeFgjEF     | 5.137855... | $x^{14} - 6x^{13} + 4x^{12} + 2x^{11} + x^{10} - x^8 + 14x^7 + \dots$           | $\{6^*, 3^8\}$      | Non-orientable |
| 12a1190 | abcdCedJEDfEgF | 4.672119... | $x^{10} - 8x^9 + 23x^8 - 47x^7 + 71x^6 - 79x^5 + \dots$                         | $\{7^*, 3^3, 4^2\}$ | Non-orientable |
| 12a1195 | abcdCeDfGFEJef | 4.807012... | $x^{12} - 7x^{11} + 11x^{10} - 12x^8 + 6x^7 + 3x^6 + \dots$                     | $\{7^*, 3^7\}$      | Non-orientable |
| 12n0037 | aBCDEFgJ       | 2.345957... | $x^8 - 5x^7 + 8x^6 - 5x^5 + 3x^4 + \dots$                                       | $\{4^*, 4^5\}$      | Orientable     |
| 12n0040 | abcDEFgJ       | 2.326729... | $x^{14} - 3x^{13} + x^{12} + 2x^{10} + 3x^9 - 2x^8 + 2x^7 + \dots$              | $\{4^*, 3^{10}\}$   | Non-orientable |
| 12n0043 | abcDEFgJ       | 2.448678... | $x^4 - 3x^3 + 2x^2 - 2x + 1$  | $\{4^*, 4^5\}$      | Orientable     |

Table C.4 – continued from previous page

|         |               |             |   |                     |                |
|---------|---------------|-------------|---|---------------------|----------------|
| 12n0103 | abcdeDfegJ    | 2.973793... | $x^{18} - 3x^{17} + x^{14} - x^{13} + 3x^{10} - 4x^9 + \dots$   | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12n0104 | abcdeDGjeF    | 2.701269... | $x^{14} - x^{13} - 3x^{12} - 3x^{11} - 2x^{10} - 2x^9 - 4x^8 - 3x^7 + \dots$                              | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12n0105 | abcdeDgjeF    | 2.874905... | $x^8 - 4x^6 - 8x^5 - 9x^4 + \dots$  | $\{14^*\}$          | Orientable     |
| 12n0106 | abcdeDFEGJ    | 2.511711... | $x^{10} - 2x^9 - 2x^8 + 3x^7 - x^6 - 5x^5 + \dots$  | $\{3^*, 3^5, 5^2\}$ | Non-orientable |
| 12n0107 | abcdedfEgJ    | 2.973793... | $x^{18} - 3x^{17} + x^{14} - x^{13} + 3x^{10} - 4x^9 + \dots$   | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12n0113 | abcdeDGIjeF   | 2.212547... | $x^{18} - x^{17} - 3x^{16} + x^{15} + 2x^{14} - 3x^{13} - 5x^{12} - x^{11} - 2x^{10} - 4x^9 + \dots$      | $\{2^*, 3^{12}\}$   | Non-orientable |
| 12n0114 | abcdeDfegIJ   | 3.272511... | $x^{18} - 4x^{17} + 2x^{16} + x^{15} + x^{14} - x^{13} + x^{12} + 4x^{10} - 8x^9 + \dots$                 | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12n0115 | abcdeDGIjef   | 1.928408... | $x^{20} - 2x^{19} + x^{18} - 3x^{17} + 2x^{16} + 3x^{14} - 2x^{13} + 3x^{12} - 5x^{11} + 3x^{10} + \dots$ | $\{1^*, 3^{13}\}$   | Non-orientable |
| 12n0150 | aBCdefgj      | 2.772695... | $x^8 - 3x^7 + 2x^6 - 5x^5 + 5x^4 + \dots$   | $\{4^*, 3^2, 4^4\}$ | Non-orientable |
| 12n0156 | abAbCDEFGJ    | 2.534503... | $x^8 - x^7 - 2x^6 - 3x^5 - 3x^4 + \dots$  | $\{4^*, 3^4, 4^3\}$ | Non-orientable |
| 12n0163 | abCdefgj      | 3.228639... | $x^8 - 3x^7 - 3x^5 + 3x^4 + \dots$  | $\{4^*, 3^2, 4^4\}$ | Non-orientable |
| 12n0172 | aBcdefgj      | 3.435339... | $x^{10} - 3x^9 - x^8 - 2x^7 + x^6 + \dots$  | $\{4^*, 3^4, 4^3\}$ | Non-orientable |
| 12n0182 | abcBcDEFGJ    | 2.675622... | $x^8 - 3x^7 + 2x^6 - 5x^5 + 7x^4 + \dots$   | $\{4^*, 3^2, 4^4\}$ | Non-orientable |
| 12n0185 | abcBdceDfegJ  | 3.177563... | $x^{18} - 3x^{17} - 2x^{16} + 4x^{15} + 3x^{14} - 3x^{13} - 4x^{12} + 4x^{11} + x^{10} - 3x^9 + \dots$    | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12n0186 | abcBdceDGjeF  | 2.950318... | $x^{16} - x^{15} - 4x^{14} - 4x^{13} - 3x^{12} - x^{11} - 2x^9 - 3x^8 + \dots$                            | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12n0187 | abcBdceDgjeF  | 3.243637... | $x^8 - 5x^6 - 12x^5 - 15x^4 + \dots$  | $\{14^*\}$          | Orientable     |
| 12n0188 | abcbdCeDfEgJ  | 2.463436... | $x^{16} - x^{15} - x^{14} - 3x^{13} - 5x^{12} - 5x^{11} - 5x^{10} - 4x^9 - 8x^8 + \dots$                  | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12n0189 | abcBdCeDFEGJ  | 2.301805... | $x^{14} - 3x^{13} + 2x^{12} - 3x^{10} + x^9 + 4x^8 - 5x^7 + \dots$  | $\{3^*, 3^7, 4^2\}$ | Non-orientable |
| 12n0190 | abcBdceDGIjeF | 2.517167... | $x^{18} - x^{17} - 5x^{16} + x^{15} + 7x^{14} - 2x^{13} - 8x^{12} + 2x^{10} - 3x^9 + \dots$               | $\{2^*, 3^{12}\}$   | Non-orientable |
| 12n0191 | abcBdceDfegIJ | 3.402739... | $x^{18} - 4x^{17} + 7x^{15} + 2x^{14} - 7x^{13} - 5x^{12} + 8x^{11} + x^{10} - 5x^9 + \dots$              | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12n0192 | abcBdceDGIjef | 2.299629... | $x^{20} - 2x^{19} - 3x^{17} + 2x^{16} + x^{15} + 5x^{14} - 2x^{13} + 3x^{12} - 5x^{11} + 3x^{10} + \dots$ | $\{1^*, 3^{13}\}$   | Non-orientable |
| 12n0233 | abcdeFGIj     | 1.645338... | $x^{20} - x^{19} - 2x^{18} + 2x^{17} + x^{16} - 2x^{15} - 2x^{14} + 2x^{12} + x^{11} - 4x^{10} + \dots$   | $\{1^*, 3^{13}\}$   | Non-orientable |
| 12n0234 | abcdefgIJ     | 2.539213... | $x^{16} - 4x^{15} + 5x^{14} - 5x^{13} + 6x^{12} - 6x^{11} + 6x^{10} - 4x^9 + 3x^8 + \dots$                | $\{2^*, 3^{12}\}$   | Non-orientable |
| 12n0235 | abcdefGIj     | 1.490735... | $x^{16} - x^{15} - x^{13} - x^{10} + x^9 + \dots$   | $\{1^*, 3^7, 5^2\}$ | Non-orientable |
| 12n0283 | aBcDCedFEGJ   | 2.567376... | $x^{12} - 5x^{11} + 8x^{10} - 4x^9 - 3x^8 + 5x^7 - 3x^6 + \dots$  | $\{5^*, 3^9\}$      | Non-orientable |
| 12n0287 | aBcDCedFgjEf  | 2.746005... | $x^{14} - 5x^{13} + 7x^{12} - 2x^{11} + x^{10} - 5x^9 - 2x^8 + 11x^7 + \dots$                             | $\{5^*, 3^9\}$      | Non-orientable |
| 12n0294 | abcdCeDfEgHj  | 2.618033... | $x^2 - 3x + \dots$  | $\{4^*, 4^5\}$      | Orientable     |
| 12n0296 | aBcDCedFEGHJ  | 2.405579... | $x^{16} - 3x^{15} + x^{14} + x^{13} - x^{12} + 2x^{11} + 2x^{10} - 2x^9 + 2x^8 + \dots$                   | $\{4^*, 3^{10}\}$   | Non-orientable |
| 12n0301 | abcDCedFEGHJ  | 2.480716... | $x^8 - 3x^7 + 2x^6 - 4x^5 + 7x^4 + \dots$   | $\{4^*, 3^2, 4^4\}$ | Non-orientable |
| 12n0387 | aBcdCedfEghj  | 3.464802... | $x^{10} - 3x^9 - x^8 - x^7 - 3x^6 - 2x^5 + \dots$   | $\{4^*, 3^{10}\}$   | Non-orientable |



Table C.4 – continued from previous page

|         |                |             |  |                     |                |
|---------|----------------|-------------|--|---------------------|----------------|
| 12n0405 | abAcbdCeDfEDgj | 3.137324... | $x^{14} - 6x^{13} + 13x^{12} - 15x^{11} + 7x^{10} + 5x^9 - 16x^8 + 21x^7 + \dots$              | $\{5^*, 3^9\}$      | Non-orientable |
| 12n0425 | aabAcBCeDGJEF  | 2.292919... | $x^{10} - x^9 - 3x^7 - 6x^6 - 3x^5 + \dots$  | $\{3^*, 3^2, 5^3\}$ | Non-orientable |
| 12n0426 | abAcbdCedfeDgj | 2.369205... | $x^4 - x^3 - 3x^2 + \dots$   | $\{14^*\}$          | Orientable     |
| 12n0428 | abAcbdCedfEgFj | 3.018654... | $x^{18} - 3x^{17} - x^{14} + x^{13} + 2x^{12} - 2x^{11} + x^{10} - x^9 + \dots$                | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12n0458 | abccDCedFEGJ   | 2.675671... | $x^{12} - 2x^{11} - 6x^9 + 2x^8 + 8x^6 + \dots$  | $\{5^*, 3^9\}$      | Non-orientable |
| 12n0459 | abcDCedFgjEf   | 2.965572... | $x^4 - 3x^3 + x^2 + \dots$   | $\{5^*, 3^9\}$      | Non-orientable |
| 12n0494 | abAbcDCeDFEGHJ | 2.669076... | $x^8 - x^7 - 2x^6 - 4x^5 - 5x^4 + \dots$   | $\{4^*, 3^{10}\}$   | Non-orientable |
| 12n0504 | abAbcbDCedFEGJ | 2.776002... | $x^{16} - 5x^{15} + 8x^{14} - 7x^{13} + 6x^{12} - x^{11} - 3x^{10} + 6x^9 - 12x^8 + \dots$     | $\{5^*, 3^9\}$      | Non-orientable |
| 12n0509 | abcbDCeDFEEgj  | 2.783559... | $x^8 - 5x^7 + 8x^6 - 7x^5 + 7x^4 + \dots$  | $\{4^*, 4^5\}$      | Orientable     |
| 12n0521 | abbcDCBeDcFEGJ | 2.846014... | $x^{16} - 5x^{15} + 9x^{14} - 11x^{13} + 10x^{12} - 6x^{11} - 2x^{10} + 14x^9 - 17x^8 + \dots$ | $\{5^*, 3^9\}$      | Non-orientable |
| 12n0531 | abcBdCeDfGjEFG | 2.536784... | $x^{16} - x^{15} - 5x^{14} + x^{13} + 6x^{12} - x^{11} - 8x^{10} + 9x^8 + \dots$               | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12n0570 | abcdedfeGFEIJ  | 1.972106... | $x^{18} - x^{17} - 2x^{16} + x^{14} - 2x^{12} - x^{11} - x^{10} + 2x^9 + \dots$                | $\{2^*, 3^{12}\}$   | Non-orientable |
| 12n0571 | abcdeDfgjEFGI  | 1.722083... | $x^4 - x^3 - x^2 + \dots$  | $\{2^*, 8^2\}$      | Orientable     |
| 12n0590 | abcdeDfEgj     | 2.799320... | $x^{18} - 3x^{17} + x^{16} - x^{15} - 3x^{13} + 4x^{12} - 2x^{11} + x^{10} + \dots$            | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12n0591 | abcdedfEgj     | 1.926788... | $x^8 - 2x^6 - 2x^5 - x^4 + \dots$  | $\{14^*\}$          | Orientable     |
| 12n0592 | abcdedFEGj     | 2.316226... | $x^8 - 2x^7 - x^6 + x^5 - x^4 + \dots$   | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12n0599 | abcbDCeDFEGJ   | 2.188056... | $x^{14} - 3x^{13} + 4x^{12} - 7x^{11} + 7x^{10} - 8x^9 + 8x^8 - 6x^7 + \dots$                  | $\{3^*, 3^7, 4^2\}$ | Non-orientable |
| 12n0603 | abcdCCedfEgj   | 2.692054... | $x^{10} - 2x^9 - 2x^8 - x^7 + 3x^6 + x^5 + \dots$  | $\{3^*, 3^5, 4^3\}$ | Non-orientable |
| 12n0609 | abcBdCeDfEgj   | 2.312226... | $x^{10} - x^9 - x^8 - 2x^7 - 4x^6 - 3x^5 + \dots$  | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12n0610 | abcdeDGJEF     | 2.141844... | $x^{10} - 2x^8 - 3x^7 - 3x^6 - 3x^5 + \dots$   | $\{3^*, 3^5, 5^2\}$ | Non-orientable |
| 12n0647 | abcBdceDfegj   | 2.598234... | $x^8 - 3x^6 - 6x^5 - 7x^4 + \dots$   | $\{14^*\}$          | Orientable     |
| 12n0653 | abcdCedFgjEf   | 3.011524... | $x^{18} - 3x^{17} - x^{15} + 2x^{14} + 3x^{13} - 2x^{12} - 3x^{11} + 3x^{10} - 2x^9 + \dots$   | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12n0657 | aabAcbdCeDFEGJ | 2.207625... | $x^{14} - 2x^{13} - 2x^{12} + 5x^{11} - x^{10} - 7x^9 + 2x^8 + 3x^7 + \dots$                   | $\{3^*, 3^{11}\}$   | Non-orientable |
| 12n0668 | abcdCedfedJEFG | 3.093323... | $x^{16} - 3x^{15} - 3x^{12} + x^{11} - 2x^9 - x^8 + \dots$                                     | $\{4^*, 3^{10}\}$   | Non-orientable |
| 12n0686 | abcdCedfGFEJef | 2.972819... | $x^{14} - 4x^{13} + 3x^{12} + 3x^{11} - 11x^{10} + 7x^9 + 6x^8 - 15x^7 + \dots$                | $\{4^*, 3^{10}\}$   | Non-orientable |
| 12n0703 | abcdcedJEDfEgF | 2.709747... | $x^{16} - 3x^{15} + 3x^{14} - 5x^{13} - x^{12} - 3x^{11} - 5x^{10} + 5x^9 - 10x^8 + \dots$     | $\{4^*, 3^{10}\}$   | Non-orientable |
| 12n0708 | abcbdcCedFEGIJ | 2.004707... | $x^{18} - x^{17} - 2x^{16} + x^{15} - x^{14} - x^{13} - 2x^{12} - x^{11} + x^{10} + \dots$     | $\{2^*, 3^{12}\}$   | Non-orientable |
| 12n0711 | abcdCeDFEGJefG | 2.637992... | $x^{16} - 3x^{15} + 4x^{13} - 3x^{12} - x^{11} - 4x^{10} - 2x^9 + 9x^8 + \dots$                | $\{4^*, 3^{10}\}$   | Non-orientable |
| 12n0754 | abcdCeDfGFEJef | 2.680021... | $x^{16} - 3x^{15} + 4x^{13} - 5x^{12} + x^{11} + 2x^{10} - 4x^9 + x^8 + \dots$                 | $\{4^*, 3^{10}\}$   | Non-orientable |
| 12n0787 | abcbdcCedFEGJ  | 2.303336... | $x^{16} - x^{15} - 2x^{14} - 4x^{12} - 3x^{11} + x^{10} - 2x^9 + \dots$                        | $\{3^*, 3^{11}\}$   | Non-orientable |

**Table C.4 – continued from previous page**

|         |                |             |   |                   |                |
|---------|----------------|-------------|---|-------------------|----------------|
| 12n0792 | abcdCCeDfEgJ   | 2.450034... | $x^{14} - 2x^{13} - 2x^{12} + 4x^{11} - x^{10} - 8x^9 - x^8 + 2x^7 + \dots$               | $\{3^*, 3^{11}\}$ | Non-orientable |
| 12n0804 | abcBdCeDfEgFeJ | 2.615554... | $x^{14} - x^{13} - 6x^{12} + 2x^{11} + 9x^{10} - 5x^9 - 4x^8 + 9x^7 + \dots$              | $\{3^*, 3^{11}\}$ | Non-orientable |
| 12n0821 | abcdedFEGIJ    | 1.803910... | $x^{18} - x^{17} - x^{16} + x^{15} - 3x^{13} - 3x^{12} - x^{11} - 2x^{10} - 4x^9 + \dots$ | $\{2^*, 3^{12}\}$ | Non-orientable |
| 12n0850 | abcdeDfegj     | 2.268444... | $x^6 - x^5 - 2x^4 - x^3 + \dots$  | $\{14^*\}$        | Orientable     |

Table C.5: Fibred knot complements with fibre  $S_{5,1}$ .

| Name    | Monodromy      | $\lambda(h)$ | $\mu_{\lambda(h)}(x)$  | Stratum                | Stable lamination |
|---------|----------------|--------------|--|------------------------|-------------------|
| 12a0146 | abcdefgHI1     | 2.068622...  | $x^{14} - 2x^{13} - x^{11} + 2x^{10} - 2x^9 + 2x^8 - x^7 + \dots$  | $\{2^*, 3^{12}, 4^2\}$ | Non-orientable    |
| 12a0576 | abcdefEgfhGIJ1 | 1.904709...  | $x^{14} - 2x^{13} + x^{10} - x^9 + x^8 - x^7 + \dots$  | $\{2^*, 3^{12}, 4^2\}$ | Non-orientable    |
| 12a0716 | abcdefghIJ     | 1.711689...  | $x^{12} - x^{11} - 2x^9 + \dots$   | $\{2^*, 3^2, 9^2\}$    | Non-orientable    |
| 12a0722 | abcdefghiJ     | 2.002893...  | $x^{10} - 3x^9 + 3x^8 - 3x^7 + 3x^6 - 3x^5 + \dots$  | $\{2^*, 10^2\}$        | Orientable        |
| 12a0835 | abcdefGHIL     | 1.770568...  | $x^{10} - 3x^9 + 4x^8 - 5x^7 + 5x^6 - 5x^5 + \dots$  | $\{2^*, 3^6, 7^2\}$    | Non-orientable    |
| 12a0838 | abcdefghiL     | 2.249025...  | $x^{18} - 3x^{17} + 2x^{16} - x^{15} + 2x^{14} - 4x^{13} + 3x^{12} - 2x^{11} + 2x^{10} - 3x^9 + \dots$                         | $\{2^*, 3^{12}, 4^2\}$ | Non-orientable    |
| 12a0850 | abcdefgfhGIL   | 2.126176...  | $x^{20} - 3x^{19} + x^{18} + 3x^{17} - 2x^{16} - 3x^{15} + 4x^{14} + 2x^{13} - 4x^{12} - 2x^{11} + 7x^{10} + \dots$            | $\{3^*, 3^{11}, 4^2\}$ | Non-orientable    |
| 12a0859 | abcdefgFhGiL   | 2.312829...  | $x^{20} - 3x^{19} + x^{18} + 3x^{17} - 4x^{16} - x^{15} + 4x^{14} - 4x^{12} + 2x^{11} + x^{10} + \dots$                        | $\{3^*, 3^{11}, 4^2\}$ | Non-orientable    |
| 12a0909 | abcdeDfEgFhGiL | 2.125964...  | $x^{22} - 2x^{21} - 2x^{20} + 3x^{19} + 3x^{18} - 3x^{17} - 3x^{16} + 4x^{15} + 4x^{14} - 3x^{13} - 2x^{12} + 2x^{11} + \dots$ | $\{3^*, 3^{15}\}$      | Non-orientable    |
| 12a1120 | abcdedfEgFHGIL | 2.086644...  | $x^{22} - 2x^{21} - 2x^{20} + 3x^{19} + 3x^{18} - x^{17} - 5x^{16} + 2x^{15} + 2x^{14} - x^{13} - 2x^{11} + \dots$             | $\{3^*, 3^{15}\}$      | Non-orientable    |
| 12a1128 | abcdefGHIJ     | 1.528388...  | $x^{12} - x^{11} - 2x^7 + \dots$   | $\{2^*, 5^2, 7^2\}$    | Non-orientable    |
| 12a1134 | abcdefgHIJ     | 1.585236...  | $x^{10} - 3x^9 + 5x^8 - 7x^7 + 7x^6 - 7x^5 + \dots$  | $\{2^*, 4^2, 8^2\}$    | Orientable        |
| 12a1273 | abcdeFGHIJ     | 1.511851...  | $x^{10} - 3x^9 + 5x^8 - 7x^7 + 9x^6 - 11x^5 + \dots$   | $\{2^*, 6^4\}$         | Orientable        |
| 12a1283 | abcdefEgFGIJJL | 1.623501...  | $x^{12} - 2x^{11} + x^{10} - x^9 + 2x^8 - x^7 - 2x^6 + \dots$  | $\{2^*, 3^8, 6^2\}$    | Non-orientable    |
| 12n0242 | abcdefghil     | 1.176280...  | $x^{10} + x^9 - x^7 - x^6 - x^5 + \dots$   | $\{18^*\}$             | Orientable        |
| 12n0574 | abcdefEgfhGijl | 1.556030...  | $x^6 - x^5 - x^4 + x^3 + \dots$  | $\{18^*\}$             | Orientable        |

## C.2 Monodromies of fibred census manifolds

The manifolds in the following tables appear in the Callahan–Hildebrand–Weeks census [24].

Table C.6: Fibred census manifolds with fibre  $S_{1,1}$ .

| Name        | Monodromy       | $\lambda(h)$ | $\mu_{\lambda(h)}(x)$ | Stratum   | Stable lamination |
|-------------|-----------------|--------------|-----------------------|-----------|-------------------|
| <i>m003</i> | <b>aaaaab</b>   | 2.618033...  | $x^2 - 3x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>m004</i> | <b>aB</b>       | 2.618033...  | $x^2 - 3x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>m009</i> | <b>aaB</b>      | 3.732050...  | $x^2 - 4x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>m010</i> | <b>aaabb</b>    | 3.732050...  | $x^2 - 4x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>m022</i> | <b>aaabAb</b>   | 4.791287...  | $x^2 - 5x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>m023</i> | <b>aaaB</b>     | 4.791287...  | $x^2 - 5x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>m039</i> | <b>aaaaB</b>    | 5.828427...  | $x^2 - 6x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>m040</i> | <b>aaabAAb</b>  | 5.828427...  | $x^2 - 6x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>m135</i> | <b>aaaabb</b>   | 5.828427...  | $x^2 - 6x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>m136</i> | <b>aaBB</b>     | 5.828427...  | $x^2 - 6x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>m206</i> | <b>aBaB</b>     | 6.854101...  | $x^2 - 7x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>m207</i> | <b>aaabbb</b>   | 6.854101...  | $x^2 - 7x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>m234</i> | <b>aaaaabb</b>  | 7.872983...  | $x^2 - 8x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>m235</i> | <b>aaaBB</b>    | 7.872983...  | $x^2 - 8x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>m369</i> | <b>aaaabbb</b>  | 9.898979...  | $x^2 - 10x + \dots$   | $\{2^*\}$ | Orientable        |
| <i>m370</i> | <b>aaBaB</b>    | 9.898979...  | $x^2 - 10x + \dots$   | $\{2^*\}$ | Orientable        |
| <i>s000</i> | <b>aaaaaB</b>   | 6.854101...  | $x^2 - 7x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>s001</i> | <b>aaabAAAb</b> | 6.854101...  | $x^2 - 7x + \dots$    | $\{2^*\}$ | Orientable        |
| <i>s298</i> | <b>aaaaBB</b>   | 9.898979...  | $x^2 - 10x + \dots$   | $\{2^*\}$ | Orientable        |
| <i>s299</i> | <b>aaaaaabb</b> | 9.898979...  | $x^2 - 10x + \dots$   | $\{2^*\}$ | Orientable        |
| <i>s463</i> | <b>aaaaabAb</b> | 10.908326... | $x^2 - 11x + \dots$   | $\{2^*\}$ | Orientable        |
| <i>s464</i> | <b>aaaBBB</b>   | 10.908326... | $x^2 - 11x + \dots$   | $\{2^*\}$ | Orientable        |
| <i>s639</i> | <b>aaaaabbb</b> | 12.922616... | $x^2 - 13x + \dots$   | $\{2^*\}$ | Orientable        |
| <i>s640</i> | <b>aaBaB</b>    | 12.922616... | $x^2 - 13x + \dots$   | $\{2^*\}$ | Orientable        |

Table C.6 – continued from previous page

|               |           |              |                     |           |            |
|---------------|-----------|--------------|---------------------|-----------|------------|
| <i>s</i> 786  | aaaabbbb  | 13.928203... | $x^2 - 14x + \dots$ | $\{2^*\}$ | Orientable |
| <i>s</i> 787  | aaBaaB    | 13.928203... | $x^2 - 14x + \dots$ | $\{2^*\}$ | Orientable |
| <i>s</i> 891  | aaaabAbb  | 14.933034... | $x^2 - 15x + \dots$ | $\{2^*\}$ | Orientable |
| <i>s</i> 892  | aaBaBB    | 14.933034... | $x^2 - 15x + \dots$ | $\{2^*\}$ | Orientable |
| <i>s</i> 960  | aaabbAbb  | 17.944271... | $x^2 - 18x + \dots$ | $\{2^*\}$ | Orientable |
| <i>s</i> 961  | aBaBaB    | 17.944271... | $x^2 - 18x + \dots$ | $\{2^*\}$ | Orientable |
| <i>v</i> 0000 | aaaaaaB   | 7.872983...  | $x^2 - 8x + \dots$  | $\{2^*\}$ | Orientable |
| <i>v</i> 0001 | aaaaBAAAB | 7.872983...  | $x^2 - 8x + \dots$  | $\{2^*\}$ | Orientable |
| <i>v</i> 0650 | aaaaaBB   | 11.916079... | $x^2 - 12x + \dots$ | $\{2^*\}$ | Orientable |
| <i>v</i> 0651 | aaaaaaabb | 11.916079... | $x^2 - 12x + \dots$ | $\{2^*\}$ | Orientable |
| <i>v</i> 1276 | aaaaBBB   | 13.928203... | $x^2 - 14x + \dots$ | $\{2^*\}$ | Orientable |
| <i>v</i> 1277 | aaaaaabAb | 13.928203... | $x^2 - 14x + \dots$ | $\{2^*\}$ | Orientable |
| <i>v</i> 1577 | aaaaBaB   | 15.937253... | $x^2 - 16x + \dots$ | $\{2^*\}$ | Orientable |
| <i>v</i> 1578 | aaaaaabb  | 15.937253... | $x^2 - 16x + \dots$ | $\{2^*\}$ | Orientable |
| <i>v</i> 2230 | aaaaabbbb | 17.944271... | $x^2 - 18x + \dots$ | $\{2^*\}$ | Orientable |
| <i>v</i> 2231 | aaBaaB    | 17.944271... | $x^2 - 18x + \dots$ | $\{2^*\}$ | Orientable |
| <i>v</i> 2898 | aaaaabAbb | 19.949874... | $x^2 - 20x + \dots$ | $\{2^*\}$ | Orientable |
| <i>v</i> 2899 | aaaBaBB   | 19.949874... | $x^2 - 20x + \dots$ | $\{2^*\}$ | Orientable |
| <i>v</i> 3246 | aaBaaBB   | 21.954451... | $x^2 - 22x + \dots$ | $\{2^*\}$ | Orientable |
| <i>v</i> 3247 | aaaabAbAb | 21.954451... | $x^2 - 22x + \dots$ | $\{2^*\}$ | Orientable |
| <i>v</i> 3510 | aaabbAbb  | 25.961481... | $x^2 - 26x + \dots$ | $\{2^*\}$ | Orientable |
| <i>v</i> 3511 | aaBaBaB   | 25.961481... | $x^2 - 26x + \dots$ | $\{2^*\}$ | Orientable |

Table C.7: Fibred census manifolds with fibre  $S_{2,1}$ .

| Name         | Monodromy | $\lambda(h)$ | $\mu_{\lambda(h)}(x)$      | Stratum      | Stable lamination |
|--------------|-----------|--------------|----------------------------|--------------|-------------------|
| <i>m</i> 036 | aaabcd    | 1.722083...  | $x^4 - x^3 - x^2 + \dots$  | $\{6^*\}$    | Orientable        |
| <i>m</i> 038 | abcdeF    | 1.722083...  | $x^4 - x^3 - x^2 + \dots$  | $\{6^*\}$    | Orientable        |
| <i>m</i> 122 | aabacdf   | 1.883203...  | $x^4 - 2x^3 + x^2 + \dots$ | $\{4^*, 4\}$ | Orientable        |

Table C.7 – continued from previous page

|              |            |             |                                   |                   |                |
|--------------|------------|-------------|-----------------------------------|-------------------|----------------|
| <i>m</i> 159 | aabACDEf   | 1.722083... | $x^4 - x^3 - x^2 + \dots$         | $\{4^*, 3^2\}$    | Non-orientable |
| <i>m</i> 160 | abCDEf     | 1.722083... | $x^4 - x^3 - x^2 + \dots$         | $\{4^*, 3^2\}$    | Non-orientable |
| <i>m</i> 184 | abaCDCEf   | 1.722083... | $x^4 - x^3 - x^2 + \dots$         | $\{3^*, 3, 4\}$   | Non-orientable |
| <i>m</i> 199 | abcdF      | 1.963553... | $x^6 - 2x^5 - x^4 + 3x^3 + \dots$ | $\{1^*, 3^5\}$    | Non-orientable |
| <i>m</i> 201 | abaCDf     | 1.722083... | $x^4 - x^3 - x^2 + \dots$         | $\{1^*, 3, 4^2\}$ | Non-orientable |
| <i>m</i> 222 | abCDf      | 1.684910... | $x^8 - x^7 - x^6 + \dots$         | $\{1^*, 3^5\}$    | Non-orientable |
| <i>m</i> 224 | abacdF     | 1.916498... | $x^8 - x^7 - x^6 - x^5 + \dots$   | $\{1^*, 3^5\}$    | Non-orientable |
| <i>m</i> 280 | aabacdcf   | 2.153721... | $x^4 - 3x^3 + 3x^2 + \dots$       | $\{4^*, 4\}$      | Orientable     |
| <i>m</i> 289 | abcD       | 2.153721... | $x^4 - 3x^3 + 3x^2 + \dots$       | $\{2^*, 4^2\}$    | Orientable     |
| <i>m</i> 304 | aabcf      | 2.081018... | $x^4 - x^3 - 2x^2 + \dots$        | $\{6^*\}$         | Orientable     |
| <i>m</i> 305 | aabcdeF    | 2.081018... | $x^4 - x^3 - 2x^2 + \dots$        | $\{6^*\}$         | Orientable     |
| <i>m</i> 336 | aabCDCEf   | 1.839286... | $x^3 - x^2 - x - 1$               | $\{3^*, 3^3\}$    | Non-orientable |
| <i>m</i> 368 | aabAACDEf  | 2.081018... | $x^4 - x^3 - 2x^2 + \dots$        | $\{4^*, 3^2\}$    | Non-orientable |
| <i>m</i> 371 | abbCDEf    | 2.081018... | $x^4 - x^3 - 2x^2 + \dots$        | $\{4^*, 3^2\}$    | Non-orientable |
| <i>m</i> 401 | aaabaacdcf | 2.015357... | $x^6 - x^5 - 4x^3 + \dots$        | $\{3^*, 3^3\}$    | Non-orientable |
| <i>s</i> 080 | abcdce     | 1.722083... | $x^4 - x^3 - x^2 + \dots$         | $\{2^*, 6\}$      | Orientable     |
| <i>s</i> 206 | abcdcef    | 1.883203... | $x^4 - 2x^3 + x^2 + \dots$        | $\{2^*, 4^2\}$    | Orientable     |
| <i>s</i> 238 | aabacdef   | 2.153721... | $x^4 - 3x^3 + 3x^2 + \dots$       | $\{2^*, 4^2\}$    | Orientable     |
| <i>s</i> 239 | aabaCDf    | 1.883203... | $x^4 - 2x^3 + x^2 + \dots$        | $\{1^*, 7\}$      | Non-orientable |
| <i>s</i> 296 | aBCDE      | 2.296630... | $x^4 - 2x^3 + \dots$              | $\{6^*\}$         | Orientable     |
| <i>s</i> 297 | aaabcbd    | 2.296630... | $x^4 - 2x^3 + \dots$              | $\{6^*\}$         | Orientable     |
| <i>s</i> 479 | aaaabcd    | 2.890053... | $x^4 - 2x^3 - 2x^2 + \dots$       | $\{6^*\}$         | Orientable     |
| <i>s</i> 519 | aabcD      | 2.296630... | $x^4 - 2x^3 + \dots$              | $\{2^*, 6\}$      | Orientable     |
| <i>s</i> 521 | aabCDf     | 1.974818... | $x^6 - 2x^5 + x^4 - 2x^3 + \dots$ | $\{1^*, 3^2, 5\}$ | Non-orientable |
| <i>s</i> 580 | aabAcf     | 2.369205... | $x^4 - x^3 - 3x^2 + \dots$        | $\{6^*\}$         | Orientable     |
| <i>s</i> 581 | aaabcdeF   | 2.369205... | $x^4 - x^3 - 3x^2 + \dots$        | $\{6^*\}$         | Orientable     |
| <i>s</i> 676 | aaaBAAcdeF | 2.369205... | $x^4 - x^3 - 3x^2 + \dots$        | $\{4^*, 3^2\}$    | Non-orientable |
| <i>s</i> 677 | abbbCDEf   | 2.369205... | $x^4 - x^3 - 3x^2 + \dots$        | $\{4^*, 3^2\}$    | Non-orientable |
| <i>s</i> 703 | abCDCf     | 1.963553... | $x^6 - 2x^5 - x^4 + 3x^3 + \dots$ | $\{1^*, 3^3, 4\}$ | Non-orientable |
| <i>s</i> 719 | aaBCD      | 3.090657... | $x^4 - 4x^3 + 4x^2 + \dots$       | $\{2^*, 4^2\}$    | Orientable     |

Table C.7 – continued from previous page

|               |             |             |                                    |                   |                |
|---------------|-------------|-------------|------------------------------------|-------------------|----------------|
| <i>s</i> 745  | aBCDCE      | 2.453170... | $x^6 - 3x^5 + 2x^4 - 2x^3 + \dots$ | $\{2^*, 3^4\}$    | Non-orientable |
| <i>s</i> 860  | aabACDf     | 2.081018... | $x^4 - x^3 - 2x^2 + \dots$         | $\{1^*, 3^2, 5\}$ | Non-orientable |
| <i>s</i> 861  | abCDCEf     | 2.081018... | $x^4 - x^3 - 2x^2 + \dots$         | $\{3^*, 3^3\}$    | Non-orientable |
| <i>s</i> 869  | aBECDCef    | 2.081018... | $x^4 - x^3 - 2x^2 + \dots$         | $\{2^*, 3^4\}$    | Non-orientable |
| <i>s</i> 911  | aaabaacdef  | 2.015357... | $x^6 - x^5 - 4x^3 + \dots$         | $\{2^*, 3^4\}$    | Non-orientable |
| <i>s</i> 912  | abCD        | 2.015357... | $x^6 - x^5 - 4x^3 + \dots$         | $\{2^*, 3^4\}$    | Non-orientable |
| <i>s</i> 923  | aaBCBDf     | 2.081018... | $x^4 - x^3 - 2x^2 + \dots$         | $\{1^*, 3, 4^2\}$ | Non-orientable |
| <i>s</i> 924  | aabcdF      | 2.369205... | $x^4 - x^3 - 3x^2 + \dots$         | $\{1^*, 3^5\}$    | Non-orientable |
| <i>s</i> 942  | aaBCDf      | 2.081018... | $x^4 - x^3 - 2x^2 + \dots$         | $\{1^*, 3^5\}$    | Non-orientable |
| <i>s</i> 943  | aabcbdF     | 2.296630... | $x^4 - 2x^3 + \dots$               | $\{1^*, 3^5\}$    | Non-orientable |
| <i>v</i> 0953 | abcdeFF     | 2.890053... | $x^4 - 2x^3 - 2x^2 + \dots$        | $\{6^*\}$         | Orientable     |
| <i>v</i> 1045 | aaabcde     | 2.296630... | $x^4 - 2x^3 + \dots$               | $\{2^*, 6\}$      | Orientable     |
| <i>v</i> 1055 | aabaabcdf   | 2.081018... | $x^4 - x^3 - 2x^2 + \dots$         | $\{3^*, 3, 4\}$   | Non-orientable |
| <i>v</i> 1076 | aaabacdf    | 2.965572... | $x^4 - 3x^3 + x^2 + \dots$         | $\{4^*, 4\}$      | Orientable     |
| <i>v</i> 1191 | abcdFF      | 2.983067... | $x^8 - 2x^7 - 3x^6 + x^4 + \dots$  | $\{1^*, 3^5\}$    | Non-orientable |
| <i>v</i> 1249 | abaCDCf     | 1.974818... | $x^6 - 2x^5 + x^4 - 2x^3 + \dots$  | $\{1^*, 3^3, 4\}$ | Non-orientable |
| <i>v</i> 1373 | aaabacdef   | 3.090657... | $x^4 - 4x^3 + 4x^2 + \dots$        | $\{2^*, 4^2\}$    | Orientable     |
| <i>v</i> 1408 | aabcdf      | 2.296630... | $x^4 - 2x^3 + \dots$               | $\{4^*, 4\}$      | Orientable     |
| <i>v</i> 1517 | abacdFF     | 2.946994... | $x^8 - 2x^7 - 2x^6 - 2x^5 + \dots$ | $\{1^*, 3^5\}$    | Non-orientable |
| <i>v</i> 1540 | aaaabcdeF   | 2.618033... | $x^2 - 3x + \dots$                 | $\{6^*\}$         | Orientable     |
| <i>v</i> 1614 | abacdcef    | 2.153721... | $x^4 - 3x^3 + 3x^2 + \dots$        | $\{2^*, 4^2\}$    | Orientable     |
| <i>v</i> 1663 | aaabacdcf   | 3.090657... | $x^4 - 4x^3 + 4x^2 + \dots$        | $\{4^*, 4\}$      | Orientable     |
| <i>v</i> 1669 | aaaaBAAcdeF | 2.618033... | $x^2 - 3x + \dots$                 | $\{4^*, 3^2\}$    | Non-orientable |
| <i>v</i> 1670 | abbbbCDEf   | 2.618033... | $x^2 - 3x + \dots$                 | $\{4^*, 3^2\}$    | Non-orientable |
| <i>v</i> 1682 | aaaaabcd    | 3.938690... | $x^4 - 3x^3 - 3x^2 + \dots$        | $\{6^*\}$         | Orientable     |
| <i>v</i> 1882 | aabcdce     | 2.296630... | $x^4 - 2x^3 + \dots$               | $\{2^*, 6\}$      | Orientable     |
| <i>v</i> 2054 | aaaBCD      | 4.056242... | $x^4 - 5x^3 + 5x^2 + \dots$        | $\{2^*, 4^2\}$    | Orientable     |
| <i>v</i> 2099 | abCDE       | 2.296630... | $x^4 - 2x^3 + \dots$               | $\{4^*, 3^2\}$    | Non-orientable |
| <i>v</i> 2101 | aaabaaccf   | 2.296630... | $x^4 - 2x^3 + \dots$               | $\{4^*, 3^2\}$    | Non-orientable |
| <i>v</i> 2272 | aabacdF     | 2.018790... | $x^6 - x^5 - 2x^4 + x^3 + \dots$   | $\{1^*, 3^3, 4\}$ | Non-orientable |

Table C.7 – continued from previous page

|            |             |             |   |                   |                |
|------------|-------------|-------------|---|-------------------|----------------|
| $v_{2345}$ | aaaabcbcd   | 3.254263... | $x^4 - 3x^3 + \dots$                      | $\{6^*\}$         | Orientable     |
| $v_{2420}$ | aabACDCEf   | 2.243329... | $x^6 - 2x^4 - 6x^3 + \dots$               | $\{3^*, 3^3\}$    | Non-orientable |
| $v_{2543}$ | abbbCDf     | 2.018790... | $x^6 - x^5 - 2x^4 + x^3 + \dots$          | $\{1^*, 3^5\}$    | Non-orientable |
| $v_{2563}$ | aabcDD      | 3.254263... | $x^4 - 3x^3 + \dots$                      | $\{2^*, 6\}$      | Orientable     |
| $v_{2587}$ | aaabacbfd   | 2.243329... | $x^6 - 2x^4 - 6x^3 + \dots$               | $\{3^*, 3^3\}$    | Non-orientable |
| $v_{2678}$ | aaabAcbd    | 3.506068... | $x^4 - 3x^3 - x^2 + \dots$                | $\{6^*\}$         | Orientable     |
| $v_{2738}$ | aabbcbdce   | 2.296630... | $x^4 - 2x^3 + \dots$                      | $\{3^*, 3, 4\}$   | Non-orientable |
| $v_{2913}$ | aaaaBaCDCEf | 2.794972... | $x^6 - 2x^5 - 6x^3 + \dots$               | $\{3^*, 3^3\}$    | Non-orientable |
| $v_{2959}$ | aabcbdf     | 2.618033... | $x^2 - 3x + \dots$                        | $\{4^*, 4\}$      | Orientable     |
| $v_{2974}$ | abacdceF    | 2.453170... | $x^6 - 3x^5 + 2x^4 - 2x^3 + \dots$        | $\{2^*, 3^4\}$    | Non-orientable |
| $v_{3007}$ | aabacdeF    | 2.453170... | $x^6 - 3x^5 + 2x^4 - 2x^3 + \dots$        | $\{2^*, 3^4\}$    | Non-orientable |
| $v_{3008}$ | aaabcd      | 2.453170... | $x^6 - 3x^5 + 2x^4 - 2x^3 + \dots$        | $\{2^*, 3^4\}$    | Non-orientable |
| $v_{3009}$ | aabAACDf    | 2.201130... | $x^6 - 2x^5 - x^4 + 2x^3 + \dots$         | $\{1^*, 3^2, 5\}$ | Non-orientable |
| $v_{3077}$ | aabcF       | 2.618033... | $x^2 - 3x + \dots$                        | $\{2^*, 4^2\}$    | Orientable     |
| $v_{3078}$ | aabaCCDf    | 2.296630... | $x^4 - 2x^3 + \dots$                      | $\{1^*, 7\}$      | Non-orientable |
| $v_{3214}$ | aaBACDEf    | 2.965572... | $x^4 - 3x^3 + x^2 + \dots$                | $\{6^*\}$         | Orientable     |
| $v_{3215}$ | aaabbcdbd   | 2.965572... | $x^4 - 3x^3 + x^2 + \dots$                | $\{6^*\}$         | Orientable     |
| $v_{3216}$ | aabcbf      | 2.890053... | $x^4 - 2x^3 - 2x^2 + \dots$               | $\{6^*\}$         | Orientable     |
| $v_{3217}$ | aBBCDE      | 2.890053... | $x^4 - 2x^3 - 2x^2 + \dots$               | $\{6^*\}$         | Orientable     |
| $v_{3261}$ | abbcdf      | 2.618033... | $x^2 - 3x + \dots$                        | $\{4^*, 4\}$      | Orientable     |
| $v_{3262}$ | aaabCDCEf   | 2.618033... | $x^2 - 3x + \dots$                        | $\{4^*, 4\}$      | Orientable     |
| $v_{3328}$ | abbCDCf     | 2.332907... | $x^6 - 2x^5 - 2x^4 + 4x^3 + \dots$        | $\{1^*, 3^3, 4\}$ | Non-orientable |
| $v_{3390}$ | aabCCDCEf   | 2.470324... | $x^6 - x^5 - x^4 - 6x^3 + \dots$          | $\{3^*, 3^3\}$    | Non-orientable |
| $v_{3411}$ | aaabacbde   | 2.296630... | $x^4 - 2x^3 + \dots$                      | $\{2^*, 3^2, 4\}$ | Non-orientable |
| $v_{3412}$ | aabCD       | 2.296630... | $x^4 - 2x^3 + \dots$                      | $\{2^*, 3^2, 4\}$ | Non-orientable |
| $v_{3423}$ | aabacbdF    | 2.326828... | $x^8 - 3x^7 + 2x^6 - 2x^5 + 3x^4 + \dots$ | $\{1^*, 3^5\}$    | Non-orientable |
| $v_{3485}$ | aaaBCBdf    | 2.369205... | $x^4 - x^3 - 3x^2 + \dots$                | $\{1^*, 3, 4^2\}$ | Non-orientable |
| $v_{3486}$ | aaabcdF     | 2.682660... | $x^8 - x^7 - 4x^6 - 2x^5 + 3x^4 + \dots$  | $\{1^*, 3^5\}$    | Non-orientable |
| $v_{3505}$ | abcDF       | 2.618033... | $x^2 - 3x + \dots$                        | $\{2^*, 4^2\}$    | Orientable     |
| $v_{3514}$ | aaaBCDf     | 2.405439... | $x^8 - x^7 - 3x^6 - 2x^5 + 4x^4 + \dots$  | $\{1^*, 3^5\}$    | Non-orientable |

Table C.7 – continued from previous page

|         |          |             |   |                |                |
|---------|----------|-------------|---|----------------|----------------|
| $v3515$ | aaabcbdf | 2.588749... | $x^8 - x^7 - 3x^6 - x^5 - 4x^4 + \dots$ | $\{1^*, 3^5\}$ | Non-orientable |
| $v3536$ | abbcdf   | 2.369205... | $x^4 - x^3 - 3x^2 + \dots$              | $\{1^*, 3^5\}$ | Non-orientable |

Table C.8: Fibred census manifolds with fibre  $S_{3,1}$ .

| Name    | Monodromy       | $\lambda(h)$ | $\mu_{\lambda(h)}(x)$                                    | Stratum             | Stable lamination |
|---------|-----------------|--------------|--|---------------------|-------------------|
| $m098$  | abcdcegh        | 1.506135...  | $x^6 - x^5 - x^3 + \dots$                                | $\{8^*, 4\}$        | Orientable        |
| $m120$  | abcBBdCefhcdeg  | 1.506135...  | $x^6 - x^5 - x^3 + \dots$                                | $\{10^*\}$          | Orientable        |
| $m146$  | aabcedh         | 1.582347...  | $x^6 - x^4 - 2x^3 + \dots$                               | $\{10^*\}$          | Orientable        |
| $m192$  | abcbdeDfEGedch  | 1.506135...  | $x^6 - x^5 - x^3 + \dots$                                | $\{4^*, 3^6\}$      | Non-orientable    |
| $m216$  | aabacbdceh      | 1.457987...  | $x^8 - x^6 - x^5 + \dots$                                | $\{7^*, 3^3\}$      | Non-orientable    |
| $m267$  | aabacbdceFGh    | 1.457987...  | $x^8 - x^6 - x^5 + \dots$                                | $\{5^*, 3^2, 5\}$   | Non-orientable    |
| $m376$  | abacbbghcbdcged | 1.582347...  | $x^6 - x^4 - 2x^3 + \dots$                               | $\{4^*, 4^3\}$      | Orientable        |
| $s148$  | aaabacbddef     | 1.465571...  | $x^3 - x^2 - 1$  | $\{8^*, 3^2\}$      | Non-orientable    |
| $s160$  | abcbdcbedcfh    | 1.457987...  | $x^8 - x^6 - x^5 + \dots$                                | $\{5^*, 3^2, 5\}$   | Non-orientable    |
| $s188$  | abacbdceGh      | 1.506135...  | $x^6 - x^5 - x^3 + \dots$                                | $\{1^*, 4^3, 5\}$   | Non-orientable    |
| $s194$  | abacbdceGh      | 1.613400...  | $x^{12} - x^{11} - x^9 - x^7 + \dots$                    | $\{1^*, 3^6, 5\}$   | Non-orientable    |
| $s309$  | aaaaabacbdcbef  | 1.465571...  | $x^3 - x^2 - 1$  | $\{6^*, 4^2\}$      | Orientable        |
| $s313$  | aaabacdef       | 1.946856...  | $x^6 - x^5 - x^4 - x^3 + \dots$                          | $\{10^*\}$          | Orientable        |
| $s363$  | abcbdccegeh     | 1.635573...  | $x^6 - 2x^5 + 2x^4 - 3x^3 + \dots$                       | $\{6^*, 3, 5\}$     | Non-orientable    |
| $s371$  | aabacbdcedcgh   | 1.506135...  | $x^6 - x^5 - x^3 + \dots$                                | $\{5^*, 4, 5\}$     | Non-orientable    |
| $s385$  | abcdEGh         | 1.532925...  | $x^{10} - x^9 - x^8 + x^5 + \dots$                       | $\{1^*, 3^5, 4^2\}$ | Non-orientable    |
| $s408$  | aabcedgh        | 1.722083...  | $x^4 - x^3 - x^2 + \dots$                                | $\{8^*, 4\}$        | Orientable        |
| $s451$  | abcdeGh         | 1.839253...  | $x^{14} - x^{13} - x^{12} - 2x^{10} + x^8 - x^7 + \dots$ | $\{1^*, 3^9\}$      | Non-orientable    |
| $s498$  | abcbdccegeh     | 1.527686...  | $x^8 - x^7 + x^6 - 3x^5 + 2x^4 + \dots$                  | $\{6^*, 3^4\}$      | Non-orientable    |
| $s526$  | abcdeF          | 2.042490...  | $x^6 - 3x^5 + 3x^4 - 3x^3 + \dots$                       | $\{2^*, 6^2\}$      | Orientable        |
| $s710$  | aabacbdcegh     | 1.610941...  | $x^8 - x^7 - 2x^5 + 2x^4 + \dots$                        | $\{6^*, 3^4\}$      | Non-orientable    |
| $s812$  | abacbdcegeh     | 1.765909...  | $x^8 - 2x^7 + 2x^6 - 4x^5 + 4x^4 + \dots$                | $\{6^*, 3^4\}$      | Non-orientable    |
| $v0521$ | abcbdcceGh      | 1.661047...  | $x^8 - 2x^7 + x^6 - x^5 + x^4 + \dots$                   | $\{1^*, 5^3\}$      | Non-orientable    |
| $v0543$ | abcdCeDfEGedch  | 1.781643...  | $x^6 - x^5 - x^4 + \dots$                                | $\{10^*\}$          | Orientable        |



Table C.8 – continued from previous page

|         |                 |             |   |                     |                |
|---------|-----------------|-------------|---|---------------------|----------------|
| $v0592$ | aabacbdcedcfh   | 1.506135... | $x^6 - x^5 - x^3 + \dots$                       | $\{5^*, 4, 5\}$     | Non-orientable |
| $v0595$ | aabacbdceGh     | 1.556030... | $x^6 - x^5 - x^4 + x^3 + \dots$                 | $\{1^*, 11\}$       | Non-orientable |
| $v0734$ | abacbdcbcdcfh   | 1.582347... | $x^6 - x^4 - 2x^3 + \dots$                      | $\{7^*, 5\}$        | Non-orientable |
| $v0789$ | abacbdcedgefh   | 1.506135... | $x^6 - x^5 - x^3 + \dots$                       | $\{4^*, 5^2\}$      | Non-orientable |
| $v0957$ | abacbdceFGh     | 1.582347... | $x^6 - x^4 - 2x^3 + \dots$                      | $\{5^*, 7\}$        | Non-orientable |
| $v1148$ | abcDEGGhChcdEF  | 1.722083... | $x^4 - x^3 - x^2 + \dots$                       | $\{4^*, 3^2, 4^2\}$ | Non-orientable |
| $v1154$ | aaabcbdef       | 2.081018... | $x^4 - x^3 - 2x^2 + \dots$                      | $\{10^*\}$          | Orientable     |
| $v1171$ | aaabacbdcbbeh   | 1.635573... | $x^6 - 2x^5 + 2x^4 - 3x^3 + \dots$              | $\{6^*, 3^2, 4\}$   | Non-orientable |
| $v1207$ | aabcedGh        | 1.582347... | $x^6 - x^4 - 2x^3 + \dots$                      | $\{1^*, 3^4, 7\}$   | Non-orientable |
| $v1302$ | aabcedF         | 2.081018... | $x^4 - x^3 - 2x^2 + \dots$                      | $\{2^*, 10\}$       | Orientable     |
| $v1638$ | abbcbdccef      | 1.671135... | $x^6 - x^5 + x^4 - 4x^3 + \dots$                | $\{8^*, 3^2\}$      | Non-orientable |
| $v1743$ | abcbdcBedcgefh  | 1.831075... | $x^6 - 2x^5 + x^3 + \dots$                      | $\{7^*, 5\}$        | Non-orientable |
| $v2063$ | abcbdccecdggh   | 1.831075... | $x^6 - 2x^5 + x^3 + \dots$                      | $\{5^*, 7\}$        | Non-orientable |
| $v2093$ | aabcbdcch       | 1.781643... | $x^6 - x^5 - x^4 + \dots$                       | $\{7^*, 5\}$        | Non-orientable |
| $v2100$ | aabaabcbdcch    | 1.722083... | $x^4 - x^3 - x^2 + \dots$                       | $\{6^*, 4^2\}$      | Non-orientable |
| $v2166$ | abcbdcch        | 1.987793... | $x^6 - 2x^4 - 3x^3 + \dots$                     | $\{10^*\}$          | Orientable     |
| $v2211$ | abcbdcceefgedch | 1.635573... | $x^6 - 2x^5 + 2x^4 - 3x^3 + \dots$              | $\{5^*, 3, 4^2\}$   | Non-orientable |
| $v2263$ | abcbdcceeggedch | 1.635573... | $x^6 - 2x^5 + 2x^4 - 3x^3 + \dots$              | $\{4^*, 3, 4, 5\}$  | Non-orientable |
| $v2296$ | aaaabcbdef      | 2.988824... | $x^6 - 2x^5 - 2x^4 - 2x^3 + \dots$              | $\{10^*\}$          | Orientable     |
| $v2334$ | aaabaabcbdccef  | 1.610941... | $x^8 - x^7 - 2x^5 + 2x^4 + \dots$               | $\{6^*, 3^4\}$      | Non-orientable |
| $v2406$ | abcBdcegh       | 1.946856... | $x^6 - x^5 - x^4 - x^3 + \dots$                 | $\{8^*, 4\}$        | Orientable     |
| $v2460$ | aaabacbdceh     | 1.873670... | $x^8 - 2x^6 - 2x^5 + \dots$                     | $\{7^*, 3^3\}$      | Non-orientable |
| $v2463$ | abcbdeefgedcfh  | 1.635573... | $x^6 - 2x^5 + 2x^4 - 3x^3 + \dots$              | $\{4^*, 3, 4, 5\}$  | Non-orientable |
| $v2474$ | abAcbdcbghcde   | 1.722083... | $x^4 - x^3 - x^2 + \dots$                       | $\{5^*, 3^2, 5\}$   | Non-orientable |
| $v2508$ | abcbdcEGh       | 1.556030... | $x^6 - x^5 - x^4 + x^3 + \dots$                 | $\{1^*, 3^3, 4^3\}$ | Non-orientable |
| $v2547$ | aaBCDEF         | 3.010783... | $x^6 - 4x^5 + 4x^4 - 4x^3 + \dots$              | $\{2^*, 6^2\}$      | Orientable     |
| $v2623$ | abcdEh          | 2.225867... | $x^6 - 3x^5 + 2x^4 - x^3 + \dots$               | $\{2^*, 4^4\}$      | Orientable     |
| $v2771$ | abacbbGhcbdcce  | 1.680261... | $x^{14} - x^{13} - 2x^{11} + x^9 - x^8 + \dots$ | $\{1^*, 3^9\}$      | Non-orientable |
| $v2815$ | abacbbfhcbdcce  | 1.582347... | $x^6 - x^4 - 2x^3 + \dots$                      | $\{2^*, 4^4\}$      | Orientable     |
| $v2834$ | aabcbdcbeh      | 1.916498... | $x^8 - x^7 - x^6 - x^5 + \dots$                 | $\{7^*, 3^3\}$      | Non-orientable |

Table C.8 – continued from previous page

|               |                |             |  |                   |                |
|---------------|----------------|-------------|--|-------------------|----------------|
| <i>v</i> 3024 | abacbdceEGh    | 1.722083... | $x^4 - x^3 - x^2 + \dots$                                | $\{1^*, 4^3, 5\}$ | Non-orientable |
| <i>v</i> 3044 | abacbdceeGh    | 1.828870... | $x^{12} - x^{11} - 2x^9 - x^7 - 2x^6 + \dots$            | $\{1^*, 3^6, 5\}$ | Non-orientable |
| <i>v</i> 3195 | abcdEGhh       | 1.661047... | $x^8 - 2x^7 + x^6 - x^5 + x^4 + \dots$                   | $\{1^*, 5^3\}$    | Non-orientable |
| <i>v</i> 3208 | aabcdceh       | 2.081018... | $x^4 - x^3 - 2x^2 + \dots$                               | $\{10^*\}$        | Orientable     |
| <i>v</i> 3304 | abcBdceDgeh    | 1.883203... | $x^4 - 2x^3 + x^2 + \dots$                               | $\{8^*, 3^2\}$    | Non-orientable |
| <i>v</i> 3320 | abcddEGh       | 1.686481... | $x^{12} - 2x^{11} + x^{10} - 2x^8 + 2x^7 - 2x^6 + \dots$ | $\{1^*, 3^9\}$    | Non-orientable |
| <i>v</i> 3327 | aaabcdeh       | 2.459633... | $x^6 - x^5 - 2x^4 - 3x^3 + \dots$                        | $\{10^*\}$        | Orientable     |
| <i>v</i> 3409 | abcbdcEDFEGGhh | 2.042490... | $x^6 - 3x^5 + 3x^4 - 3x^3 + \dots$                       | $\{8^*, 3^2\}$    | Non-orientable |
| <i>v</i> 3418 | abbcbdceh      | 1.781643... | $x^6 - x^5 - x^4 + \dots$                                | $\{7^*, 3^3\}$    | Non-orientable |

Table C.9: Fibred census manifolds with fibre  $S_{4,1}$ .

| Name          | Monodromy       | $\lambda(h)$ | $\mu_{\lambda(h)}(x)$                             | Stratum             | Stable lamination |
|---------------|-----------------|--------------|---|---------------------|-------------------|
| <i>m</i> 147  | aabcdefgj       | 1.457987...  | $x^8 - x^6 - x^5 + \dots$                         | $\{14^*\}$          | Orientable        |
| <i>m</i> 150  | abcdefgij       | 1.280638...  | $x^8 - x^5 - x^4 + \dots$                         | $\{12^*, 4\}$       | Orientable        |
| <i>m</i> 231  | abcdcedfegj     | 1.331136...  | $x^{10} - x^7 - x^6 + \dots$                      | $\{11^*, 3^3\}$     | Non-orientable    |
| <i>m</i> 389  | abcdCedfEghj    | 1.401268...  | $x^6 - x^4 - x^3 + \dots$                         | $\{14^*\}$          | Orientable        |
| <i>s</i> 021  | abcdcedfegijefg | 1.280638...  | $x^8 - x^5 - x^4 + \dots$                         | $\{7^*, 9\}$        | Non-orientable    |
| <i>s</i> 235  | abcbdcdfedjefg  | 1.324717...  | $x^3 - x - 1$                                     | $\{8^*, 4^3\}$      | Orientable        |
| <i>s</i> 247  | abcdedfegij     | 1.359999...  | $x^8 - x^7 + x^6 - 2x^5 + x^4 + \dots$            | $\{10^*, 3, 5\}$    | Non-orientable    |
| <i>s</i> 483  | aabcdedfegj     | 1.556030...  | $x^6 - x^5 - x^4 + x^3 + \dots$                   | $\{11^*, 5\}$       | Non-orientable    |
| <i>s</i> 692  | aabacbdcedfegj  | 1.401268...  | $x^6 - x^4 - x^3 + \dots$                         | $\{9^*, 3^3, 4\}$   | Non-orientable    |
| <i>v</i> 0011 | abcdcedfeghjefg | 1.280638...  | $x^8 - x^5 - x^4 + \dots$                         | $\{9^*, 7\}$        | Non-orientable    |
| <i>v</i> 0321 | abacbdcedfeGIj  | 1.401268...  | $x^6 - x^4 - x^3 + \dots$                         | $\{1^*, 4^4, 7\}$   | Non-orientable    |
| <i>v</i> 0329 | abacbdcedfegIj  | 1.472353...  | $x^{10} - x^9 - x^6 + \dots$                      | $\{1^*, 3^8, 7\}$   | Non-orientable    |
| <i>v</i> 0802 | abcbdcdfegij    | 1.425005...  | $x^8 - x^7 - x^5 + x^4 + \dots$                   | $\{8^*, 3, 7\}$     | Non-orientable    |
| <i>v</i> 0810 | abcbdcdfegfeij  | 1.523060...  | $x^8 - x^7 - x^6 + x^4 + \dots$                   | $\{7^*, 9\}$        | Non-orientable    |
| <i>v</i> 0960 | abcdefGIj       | 1.490735...  | $x^{16} - x^{15} - x^{13} - x^{10} + x^9 + \dots$ | $\{1^*, 3^7, 5^2\}$ | Non-orientable    |
| <i>v</i> 0984 | aaabcdefgh      | 1.987793...  | $x^6 - 2x^4 - 3x^3 + \dots$                       | $\{14^*\}$          | Orientable        |

Table C.9 – continued from previous page

|         |                |             |  |                     |                |
|---------|----------------|-------------|--|---------------------|----------------|
| $v1217$ | abcdefgH       | 2.011287... | $x^8 - 3x^7 + 3x^6 - 3x^5 + 3x^4 + \dots$  | $\{2^*, 8^2\}$      | Orientable     |
| $v1811$ | abcdedfegIj    | 1.677784... | $x^{12} - 2x^{11} + x^{10} - x^9 + x^7 - x^6 + \dots$                            | $\{1^*, 3^5, 4^4\}$ | Non-orientable |
| $v1845$ | abcdCedfegij   | 1.506135... | $x^6 - x^5 - x^3 + \dots$  | $\{10^*, 3^4\}$     | Non-orientable |
| $v1935$ | abcdefgIj      | 1.812464... | $x^{20} - x^{19} - x^{18} - x^{16} - 2x^{14} + x^{12} - x^{11} + x^{10} + \dots$ | $\{1^*, 3^{13}\}$   | Non-orientable |
| $v1962$ | abacbdcedfegij | 1.506135... | $x^6 - x^5 - x^3 + \dots$  | $\{8^*, 3^4, 4\}$   | Non-orientable |
| $v2092$ | aabcdefgij     | 1.582347... | $x^6 - x^4 - 2x^3 + \dots$   | $\{12^*, 4\}$       | Orientable     |
| $v2381$ | abcdbccedfegj  | 1.556030... | $x^6 - x^5 - x^4 + x^3 + \dots$  | $\{9^*, 3^5\}$      | Non-orientable |
| $v2552$ | aBCDEFGJ       | 2.060178... | $x^8 - 3x^7 + 2x^6 + x^5 - 3x^4 + \dots$   | $\{2^*, 6^3\}$      | Orientable     |
| $v2585$ | abcdefegj      | 1.781643... | $x^6 - x^5 - x^4 + \dots$  | $\{14^*\}$          | Orientable     |
| $v3101$ | abcBdcedfegj   | 1.590980... | $x^{10} - 2x^7 - 2x^6 + \dots$   | $\{11^*, 3^3\}$     | Non-orientable |
| $v3248$ | aaabcdefgj     | 2.145106... | $x^8 - x^7 - 2x^6 - x^5 + x^4 + \dots$   | $\{14^*\}$          | Orientable     |

Table C.10: Fibred census manifolds with fibre  $S_{5,1}$ .

| Name    | Monodromy       | $\lambda(h)$ | $\mu_{\lambda(h)}(x)$                    | Stratum         | Stable lamination |
|---------|-----------------|--------------|--|-----------------|-------------------|
| $m016$  | abcdefghil      | 1.176280...  | $x^{10} + x^9 - x^7 - x^6 - x^5 + \dots$ | $\{18^*\}$      | Orientable        |
| $m169$  | abcdefgfhgil    | 1.216391...  | $x^{10} - x^6 - x^5 + \dots$             | $\{15^*, 5\}$   | Non-orientable    |
| $m303$  | aabcdefghil     | 1.401268...  | $x^6 - x^4 - x^3 + \dots$                | $\{18^*\}$      | Orientable        |
| $s063$  | abcdedfegfhgil  | 1.261230...  | $x^{10} - x^8 - x^5 + \dots$             | $\{13^*, 7\}$   | Non-orientable    |
| $v0730$ | abcdCedfegfhgil | 1.352049...  | $x^{14} - x^{12} - x^{11} + \dots$       | $\{13^*, 3^5\}$ | Non-orientable    |
| $v1812$ | abcdefghikl     | 1.401268...  | $x^6 - x^4 - x^3 + \dots$                | $\{16^*, 4\}$   | Orientable        |
| $v2407$ | aabcdefgfhgil   | 1.472353...  | $x^{10} - x^9 - x^6 + \dots$             | $\{15^*, 5\}$   | Non-orientable    |
| $v2704$ | abcdCedfEgfhgil | 1.431000...  | $x^{10} - x^9 - x^8 + x^7 - x^5 + \dots$ | $\{15^*, 3^3\}$ | Non-orientable    |

# Bibliography

- [1] Alejandro Adem, Johann Leida, and Yongbin Ruan. *Orbifolds and Stringy Topology*. Cambridge University Press, 2007. Cambridge Books Online. [52]
- [2] S. I. Adyan. Algorithmic unsolvability of problems of recognition of certain properties of groups. *Dokl. Akad. Nauk SSSR (N.S.)*, 103:533–535, 1955. [48]
- [3] Ian Agol. Ideal triangulations of pseudo-Anosov mapping tori. In *Topology and geometry in dimension three*, volume 560 of *Contemp. Math.*, pages 1–17. Amer. Math. Soc., Providence, RI, 2011. [32, 33, 34, 35, 57, 68]
- [4] Ian Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With an appendix by Agol, Daniel Groves, and Jason Manning. [4]
- [5] Ian Agol, Joel Hass, and William Thurston. The computational complexity of knot genus and spanning area. *Trans. Amer. Math. Soc.*, 358(9):3821–3850, 2006. [12, 13, 55]
- [6] Erwin H. Bareiss. Sylvester’s identity and multistep integer-preserving Gaussian elimination. *Math. Comp.*, 22:565–578, 1968. [68]
- [7] Saugata Basu, Richard Pollack, and Marie-Françoise Roy. *Algorithms in real algebraic geometry*, volume 10 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, second edition, 2006. [39]
- [8] M. C. Bell. An algorithm for deciding reducibility. *ArXiv e-prints*, March 2014. [xii, 19, 22, 27]
- [9] M. C. Bell. Recognising mapping classes. *ArXiv e-prints*, October 2014. [xii, 28, 48]
- [10] Mark Bell. Table of Bundles. Available at [https://bitbucket.org/Mark\\_Bell/bundle-censuses](https://bitbucket.org/Mark_Bell/bundle-censuses). [88]

- [11] Mark Bell. flipper (computer software). [https://bitbucket.org/Mark\\_Bell/flipper/](https://bitbucket.org/Mark_Bell/flipper/), 2013–2015. Version 0.9.7. [67]
- [12] Diego Benardete, Mauricio Gutiérrez, and Zbigniew Nitecki. A combinatorial approach to reducibility of mapping classes. In *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*, volume 150 of *Contemp. Math.*, pages 1–31. Amer. Math. Soc., Providence, RI, 1993. [4]
- [13] M. Bestvina and M. Handel. Train-tracks for surface homeomorphisms. *Topology*, 34(1):109–140, 1995. [19]
- [14] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups. *Ann. of Math. (2)*, 135(1):1–51, 1992. [19]
- [15] Joan Birman, Ki Hyoung Ko, and Sang Jin Lee. A new approach to the word and conjugacy problems in the braid groups. *Adv. Math.*, 139(2):322–353, 1998. [4]
- [16] Joan S. Birman. *Braids, links, and mapping class groups*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 82. [4]
- [17] O. Bogopolski, A. Martino, O. Maslakova, and E. Ventura. The conjugacy problem is solvable in free-by-cyclic groups. *Bull. London Math. Soc.*, 38(5):787–794, 2006. [52]
- [18] Endre Boros, Khaled Elbassioni, Vladimir Gurvich, and Kazuhisa Makino. Generating vertices of polyhedra and related problems of monotone generation. In *Polyhedral computation*, volume 48 of *CRM Proc. Lecture Notes*, pages 15–43. Amer. Math. Soc., Providence, RI, 2009. [20]
- [19] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. [12]
- [20] Peter Brinkmann. An implementation of the Bestvina-Handel algorithm for surface homeomorphisms. *Experiment. Math.*, 9(2):235–240, 2000. [69]
- [21] Peter Brinkmann. Xtrain (computer software). <https://gitorious.org/xtrain>, 2009. [69, 81]

- [22] Gerhard Burde, Heiner Zieschang, and Michael Heusener. *Knots*, volume 5 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, extended edition, 2014. [3, 48, 71]
- [23] Benjamin A. Burton, Ryan Budney, William Pettersson, et al. Regina: Software for 3-manifold topology and normal surface theory. <http://regina.sourceforge.net/>, 1999–2014. [70, 72]
- [24] Patrick J. Callahan, Martin V. Hildebrand, and Jeffrey R. Weeks. A census of cusped hyperbolic 3-manifolds. *Math. Comp.*, 68(225):321–332, 1999. With microfiche supplement. [88, 117]
- [25] Matthieu Calvez. Fast Nielsen-Thurston classification of braids. *Algebr. Geom. Topol.*, 14(3):1745–1758, 2014. [4, 5]
- [26] Matthieu Calvez and Bert Wiest. Fast algorithmic Nielsen-Thurston classification of four-strand braids. *J. Knot Theory Ramifications*, 21(5):1250043, 25, 2012. [4]
- [27] Andrew J. Casson and Steven A. Bleiler. *Automorphisms of surfaces after Nielsen and Thurston*, volume 9 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1988. [28, 30]
- [28] J. C. Cha and C. Livingston. KnotInfo: Table of Knot Invariants. Available at <http://www.indiana.edu/~knotinfo>. [88, 90]
- [29] Henri Cohen. *A course in computational algebraic number theory*, volume 138 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, 1993. [38]
- [30] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to algorithms*. MIT Press, Cambridge, MA; McGraw-Hill Book Co., Boston, MA, second edition, 2001. [17]
- [31] Marc Culler, Nathan M. Dunfield, and Jeffrey R. Weeks. SnapPy, a computer program for studying the topology of 3-manifolds. Available at <http://snappy.computop.org> (20/09/2014). [70]
- [32] M. Dehn. Über unendliche diskontinuierliche Gruppen. *Math. Ann.*, 71(1):116–144, 1911. [48]
- [33] M. Dehn. Die Gruppe der Abbildungsklassen. *Acta Math.*, 69(1):135–206, 1938. Das arithmetische Feld auf Flächen. [1]

- [34] Max Dehn. *Papers on group theory and topology*. Springer-Verlag, New York, 1987. Translated from the German and with introductions and an appendix by John Stillwell, With an appendix by Otto Schreier. [48]
- [35] Jeff Erickson and Amir Nayyeri. Tracing compressed curves in triangulated surfaces. *Discrete Comput. Geom.*, 49(4):823–863, 2013. [12, 13, 25, 53]
- [36] Benson Farb, editor. *Problems on mapping class groups and related topics*, volume 74 of *Proceedings of Symposia in Pure Mathematics*. American Mathematical Society, Providence, RI, 2006. [3, 48]
- [37] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012. [1, 2, 3, 6, 13, 18, 19, 49, 50, 57, 63, 66]
- [38] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012. [27, 76]
- [39] Albert Fathi, François Laudenbach, and Valentin Poénaru. *Thurston’s work on surfaces*, volume 48 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 2012. Translated from the 1979 French original by Djun M. Kim and Dan Margalit. [6, 30, 76]
- [40] Laurent Fousse, Guillaume Hanrot, Vincent Lefèvre, Patrick Pélissier, and Paul Zimmermann. MPFR: A multiple-precision binary floating-point library with correct rounding. *ACM Transactions on Mathematical Software*, 33(2):13:1–13:15, June 2007. [69]
- [41] George K. Francis. *A topological picturebook*. Springer-Verlag, New York, 1987. [3]
- [42] D. J. H. Garling. *Inequalities: a journey into linear analysis*. Cambridge University Press, Cambridge, 2007. [20]
- [43] F. A. Garside. The braid group and other groups. *Quart. J. Math. Oxford Ser. (2)*, 20:235–254, 1969. [4]
- [44] Toby Hall. Trains (computer software). [http://www.liv.ac.uk/~tobyhall/T\\_Hall.html](http://www.liv.ac.uk/~tobyhall/T_Hall.html). [69, 81]

- [45] Hessam Hamidi-Tehrani. *Algorithms in the surface mapping class groups*. ProQuest LLC, Ann Arbor, MI, 1997. Thesis (Ph.D.)—Columbia University. [65, 66, 67]
- [46] W. Hart, F. Johansson, and S. Pancratz. FLINT: Fast Library for Number Theory, 2013. Version 2.4.0, <http://flintlib.org>. [69]
- [47] Allen Hatcher. On triangulations of surfaces. *Topology Appl.*, 40(2):189–194, 1991. [11]
- [48] Lane A. Hemaspaandra and Mitsunori Ogihara. *The complexity theory companion*. Texts in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, 2002. [8, 62]
- [49] Geoffrey Hemion. On the classification of homeomorphisms of 2-manifolds and the classification of 3-manifolds. *Acta Math.*, 142(1-2):123–155, 1979. [48]
- [50] Jim Hoste, Morwen Thistlethwaite, and Jeff Weeks. The first 1,701,936 knots. *Math. Intelligencer*, 20(4):33–48, 1998. [88, 90]
- [51] A. Ya. Khinchin. *Continued fractions*. The University of Chicago Press, Chicago, Ill.-London, 1964. [2]
- [52] Yoshikata Kida. The mapping class group from the viewpoint of measure equivalence theory. *Mem. Amer. Math. Soc.*, 196(916):viii+190, 2008. [27]
- [53] Donald E. Knuth. *The art of computer programming. Vol. 2*. Addison-Wesley, Reading, MA, 1998. Seminumerical algorithms, Third edition [of MR0286318]. [46]
- [54] Thomas Koberda and Johanna Mangahas. An effective algebraic detection of the Nielsen-Thurston classification of mapping classes. *J. Topol. Anal.*, 7(1):1–21, 2015. [4, 19, 27]
- [55] Catherine Labruère and Luis Paris. Presentations for the punctured mapping class groups in terms of Artin groups. *Algebr. Geom. Topol.*, 1:73–114 (electronic), 2001. [viii, 7]
- [56] A. K. Lenstra, H. W. Lenstra, Jr., and L. Lovász. Factoring polynomials with rational coefficients. *Math. Ann.*, 261(4):515–534, 1982. [38]
- [57] Gilbert Levitt and Jean-Louis Nicolas. On the maximum order of torsion elements in  $GL(n, \mathbf{Z})$  and  $\text{Aut}(F_n)$ . *J. Algebra*, 208(2):630–642, 1998. [2]



- [58] Jérôme E. Los. Pseudo-Anosov maps and invariant train tracks in the disc: a finite algorithm. *Proc. London Math. Soc. (3)*, 66(2):400–430, 1993. [4]
- [59] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.*, 10(4):902–974, 2000. [4, 5, 6, 26, 34]
- [60] Howard Masur, Lee Mosher, and Saul Schleimer. On train-track splitting sequences. *Duke Math. J.*, 161(9):1613–1656, 2012. [28]
- [61] Yukio Matsumoto and José María Montesinos-Amilibia. Pseudo-periodic homeomorphisms and degeneration of Riemann surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 30(1):70–75, 1994. [58]
- [62] Sergei Matveev. *Algorithmic topology and classification of 3-manifolds*, volume 9 of *Algorithms and Computation in Mathematics*. Springer, Berlin, second edition, 2007. [11]
- [63] Curtis T. McMullen. Teichmüller Theory Notes, 2005. [28]
- [64] W. Menasco and J. Ringland. Bh (computer software). <http://copper.math.buffalo.edu/BH/index.html>, 1999–2011. [69, 81]
- [65] Christof Menzel and John R. Parker. Pseudo-Anosov diffeomorphisms of the twice punctured torus. In *Recent advances in group theory and low-dimensional topology (Pusan, 2000)*, volume 27 of *Res. Exp. Math.*, pages 141–154. Heldermann, Lemgo, 2003. [30, 71]
- [66] Lee Mosher. The classification of pseudo-Anosovs. In *Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984)*, volume 112 of *London Math. Soc. Lecture Note Ser.*, pages 13–75. Cambridge Univ. Press, Cambridge, 1986. [48]
- [67] Lee Mosher. Tiling the projective foliation space of a punctured surface. *Trans. Amer. Math. Soc.*, 306(1):1–70, 1988. [30]
- [68] Lee Mosher. Mapping class groups are automatic. *Ann. of Math. (2)*, 142(2):303–384, 1995. [50, 62]
- [69] Lee Mosher. Train track expansions of measured foliations. Available at <http://andromeda.rutgers.edu/~mosher/>, 2003. Preprint. [60]

- [70] Lee Mosher. Mapping class groups. Available at <http://andromeda.rutgers.edu/~mosher/>, 2007. Lecture notes. [50, 58, 59, 60]
- [71] Udo Pachner. P.L. homeomorphic manifolds are equivalent by elementary shellings. *European J. Combin.*, 12(2):129–145, 1991. [11]
- [72] Athanase Papadopoulos. Introduction to Teichmüller theory, old and new. In *Handbook of Teichmüller theory. Vol. I*, volume 11 of *IRMA Lect. Math. Theor. Phys.*, pages 1–30. Eur. Math. Soc., Zürich, 2007. [28, 29]
- [73] J. R. Parker and C. Series. The mapping class group of the twice punctured torus. In *Groups: topological, combinatorial and arithmetic aspects*, volume 311 of *London Math. Soc. Lecture Note Ser.*, pages 405–486. Cambridge Univ. Press, Cambridge, 2004. [71]
- [74] R. C. Penner. Bounds on least dilatations. *Proc. Amer. Math. Soc.*, 113(2):443–450, 1991. [75]
- [75] R. C. Penner and J. L. Harer. *Combinatorics of train tracks*, volume 125 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992. [28, 31, 32]
- [76] Robert C. Penner. The action of the mapping class group on curves in surfaces. *Enseign. Math. (2)*, 30(1-2):39–55, 1984. [76]
- [77] Robert C. Penner. An introduction to train tracks. In *Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984)*, volume 112 of *London Math. Soc. Lecture Note Ser.*, pages 77–90. Cambridge Univ. Press, Cambridge, 1986. [31]
- [78] Michael O. Rabin. Recursive unsolvability of group theoretic problems. *Ann. of Math. (2)*, 67:172–194, 1958. [48]
- [79] Tibor Radó. Über den begriff der riemannschen fläche. *Acta Litt. Sci. Szeged*, 2:101–121, 1925. [10]
- [80] Marcus Schaefer, Eric Sedgwick, and Daniel Štefankovič. Algorithms for normal curves and surfaces. In *Computing and combinatorics*, volume 2387 of *Lecture Notes in Comput. Sci.*, pages 370–380. Springer, Berlin, 2002. [12, 13]
- [81] Marcus Schaefer, Eric Sedgwick, and Daniel Štefankovic. Computing Dehn twists and geometric intersection numbers in polynomial time, 2007. [13, 69]

- [82] Michael Sipser. *Introduction to the Theory of Computation*. International Thomson Publishing, 1st edition, 1996. [8, 78, 79]
- [83] W. A. Stein et al. *Sage Mathematics Software (Version 6.5)*. The Sage Development Team, 2014. <http://www.sagemath.org>. [69]
- [84] Jing Tao. Linearly bounded conjugator property for mapping class groups. *Geom. Funct. Anal.*, 23(1):415–466, 2013. [4, 5, 34, 48]
- [85] The PARI Group, Bordeaux. *PARI/GP version 2.7.0*, 2014. available from <http://pari.math.u-bordeaux.fr/>. [69]
- [86] William P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982. [67]
- [87] William P Thurston and John Willard Milnor. *The geometry and topology of three-manifolds*. Princeton University Princeton, 1979. [3, 28]
- [88] Michel Waldschmidt. *Diophantine approximation on linear algebraic groups*, volume 326 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2000. Transcendence properties of the exponential function in several variables. [39]